UNIVERSITY OF LIVERPOOL

Loop Study of Gribov-Zwanziger Confinement and Mass Operators in Quantum Chromodynamics.

Thesis submitted in accordance with the requirements of

the University of Liverpool for the degree of Doctor in Philosophy

by

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August 2009

Acknowledgements

As is customary, I will begin by thanking all of the people that made any sort of contribution towards this thesis. Ben, Cathy, Chris, Elisa, Kirk and Rob, occasional, it has always been my preference to work from home, office colleagues assisting with countless tasks. In particular, Ben and Chris for their patience when introducing a complete novice to the UNIX operating system. Alon, Thomas (Teubner) and Radu for the postgraduate lecture courses they delivered during my first year. Steve for turning my Dell laptop into a tool for performing loop calculations. John for so many things, mostly patience and the fact that his door was always open. The University of Liverpool for a research studentship. Finally, special thanks go to my Mum, Liz, and Dad, Bryce, not least for the fact that they put a roof over my head and gave me pocket money during the final three months of this work.

Abstract

We consider separately the Gribov-Zwanziger approach to *global* gauge fixing and gauge invariant dynamical mass operators in QCD. The Gribov-Zwanziger study focuses on the possible implications for colour confinement of gluons by studying the gap equation and Faddeev-Popov ghost propagator in the presence of massive quarks, and comparing the result with the Kugo-Ojima confinement criterion. This is done to two loop order in the $\overline{\text{MS}}$ scheme. In the dynamical mass operator study, we consider an approximation to the (non-local) gauge invariant mass operator A^2_{min}. The non-local aproximation $F^a_{\mu\nu}[(D^2)^{-1}]^{ab}F^b_{\mu\nu}$ is also gauge invariant and may be cast into local form and incorporated into a renormalizable action. Using the local renormalizable action, we calculate the anomalous dimension for the mass operatore to three loop order. Whilst each subject is discussed separately, the issues of gauge fixing and mass generation in a non-Abelian gauge theory include many similarities, in particular the occurrence of a non-locality. Tackling this issue in each area appears to proceed in a relatively straightforward manner using standard techniques. However, the resulting local theories are complicated by the inclusion of additional parameters which include Bose and Grassmann fields, mass objects and couplings. This makes it very difficult to derive an effective action using, for example, the local composite operator formalism, application of which is also known to introduce additional complexities.

Original work presented in this thesis is published in
F.R. Ford & J.A. Gracey, J. Phys. **A42** (2009), 325402.
F.R. Ford & J.A. Gracey, Phys. Lett. **B674** (2009), 232.

Contents

Chapter 1

Introduction

When Yang-Mills theory, [1],[2], was shown to be renormalizable, even after symmetry breaking, [3],[4],[5], a quantum theoretical interpretation of particle interactions, using a path integral approach to gauge theories, became a genuine possibility. The difficulties that were overcome are conveniently introduced by noting that, in principle, the path integral of Maxwell's theory of electromagnetism is undefined because of the gauge degree of freedom. Since the path integral measure $\mathcal{D}A$ and the action S are both gauge invariant, this means that functionally integrating over $\mathcal{D}A$ will eventually over-count the degrees of freedom of the theory. The problem is that of selecting a single "slice" from the gauge orbit in order to avoid infinite over-counting, the gauge fixing problem. Inserting a delta function into the functional integration $\mathcal{D}A$ changes the measure of integration. In order to achieve this in an unambiguous manner, we must select a specific gauge configuration by inserting a delta function into the path integral using an expression which is formally equal to the number 1. This is the well known Faddeev-Popov gauge fixing procedure, [2], that introduces quite naturally the extra term, identified by Richard Feynman [6], necessary to solve the unitarity issues associated with a naive quantization of Yang-Mills theory.

The theoretical breakthrough was required to explain the discovery of three identical sets of lepton pairs, corresponding to the electron, muon, and τ particles and their corresponding neutrino partners. It is used to describe the Weinberg-Salam model, [7],[8], a curious amalgam of the weak, Yang-Mills type SU(2), and electromagnetic, Maxwell type U(1), interactions. The complete action consists of three different pieces; a gauge part, a fermionic part, and a scalar Higgs sector included to induce symmetry breaking. After symmetry breaking, the SU(2) and U(1) gauge fields recombine and emerge as the physical photon field, a neutral massive vector particle and a charged doublet of vector particles. The Weinberg-Salam model is capable of describing processes such as lepton decay to the lowest order. Quantization of the Weinberg-Salam model is complicated by the seemingly impossible task of simultaneously retaining the features of unitarity and renormalizability in a spontaneously broken gauge theory with massive propagators. To show that spontaneously broken gauge theories can be both unitary and renormalizable, it is necessary to appeal to a special type of gauge R_ξ, [5], in which it is possible to interpolate between two sets of propagators. In particular a gauge fixing term proportional to α is inserted into the Lagrangian such that in the limit $\alpha \to 0$ the theory is described by a propagator that is $O(1/k^2)$ in the ultraviolet limit, this results in good power counting behaviour in Feynman graphs, and so the theory appears to be renormalizable but not unitary. In the limit $\alpha \to \infty$ the theory is described by a propagator that behaves as a constant in the ultraviolet limit, which is disastrous from the perspective of renormalizability, but, from the S matrix point of view the theory is unitary. Although the Green's functions of the theory are dependent on α, we know that the S matrix is not and so in some sense it is possible to consider that the quantized Weinberg-Salam theory is both renormalizable and unitary.

Hadrons are bound states of strongly interacting particles and are present in ordinary matter in the form of protons and neutrons. In 1947 new types of hadrons not present in ordinary matter were discovered in cosmic ray experiments and by the early 1960's several dozens of different types of hadrons were known to exist from resonances found in scattering experiments. A new theoretical framework was required to interpret this multitude of states. The quark model grew out of a realization by Gell-Mann *et al*, [9], that all of the recently observed hadrons could be simply interpreted as bound states of just three fundamental spin-$\frac{1}{2}$ particles and their antiparticles. The triplet of particles transformed in the fundamental representation **3** of SU(3). The simple quark model assumes that only three types of quark bound states are allowed. These are, the *baryons* which have half-integer spin and are assumed to be bound states of three quarks ($3q$); the *anti-baryons*, which are there antiparticles and are assumed to be bound states of three anti-quarks ($3\bar{q}$); and the *mesons*, which have integer spin and are assumed to be bound states of a quark and an anti-quark ($q\bar{q}$). The original three quarks have been expanded to six quarks: up, down, strange, charm, bottom and top. The global symmetry group SU(N_f) for N_f quarks is now called the "flavour" symmetry. Although the quark model enjoyed great success bringing order to the chaos of the hundreds of resonances found in scattering experiments, the questions remained: why were the quarks not observed experimentally? Were they real, or were they just a useful mathematical device? Quantum Chromodynamics (QCD) is the leading candidate for a theory capable of answering these questions. QCD describes the strong interaction in terms of a Yang-Mills gauge field carrying an SU(3) "colour" force where the quarks carry two indices, the flavour index, a global symmetry which is not gauged, and the colour index participating in the local gauge symmetry. From the perspective of QCD, the flavour index is relegated to a minor role with respect to the colour force which is responsible for binding the quarks together.

The great success of QCD as a theoretical description of the strong interaction came in the form of the renormalization group equations (RGE's), [10],[11]. By imposing deceptively simple constraints on the renormalized vertex functions of a renormalizable field theory, RGE's are capable of yielding non-trivial consequences, most notably the property of asymptotic freedom, [12],[13]. The simple constraints are formulated in terms of the observation that the physical content of the theory cannot depend on the subtraction point μ which has only been introduced as a purely mathematical device for use in the regularization and renormalization process. That is, if the subtraction point μ is changed, other parameters such as the masses and the coupling constants, must also change in order to compensate for this effect. The significant breakthrough was the discovery that the high energy behaviour seen at the SLAC experiments could be explained by the property of asymptotic freedom identified in non-Abelian gauge theory using RGE's. A change in subtraction point μ (*energy*) and the resulting change in parameter is associated with the scaling phenomena observed in the deep inelastic scattering experiments performed at SLAC. The simplest explanation of scaling is given by Feynman's parton model, where the proton is considered to consist of point like constituents. The success of the parton model in explaining the scaling behaviour observed at SLAC led to confusion. If the proton was a bound state of some mysterious force, then presumably nonperturbative effects were dominant. However, the parton model indicated that, at high energies, the partons (e.g. quarks) could be considered to act like free point particles. Apparently, nonperturbative effects could somehow be neglected, and we could assume that quarks were free to roam inside the proton. This is precisely the picture of asymptotic freedom in the ultraviolet sector predicted by the RGE's of QCD.

In spite of being the single candidate for a theory of the strong interactions, QCD fails to reproduce many of the essential low energy features of the hadron world, such as the spectrum of low lying hadron states. Perturbation theory is only effective in the high energy, asymptotically free, region where we can use the renormalization group equations to make comparisons between theory and experimental data. Consideration of nonperturbative effects in quantum field theory is notoriously difficult. Methods include Dyson Schwinger equation (DSE) methods, [14],[15], and Wilson's lattice gauge theory, [16]. Lattice gauge theory in particular provides compelling evidence in support of the confinement picture which explains why quarks are not observed in isolation. In principle, if the potential between two quarks is proportional to the distance between them, then two quarks can never be separated. Lattice gauge theory provides a vivid description of colour confinement in terms of local gauge invariance. Local gauge invariance dictates that coloured, and hence gauge variant, quark states are always accompanied by a 'string' of gluons to form a colour singlet locally at any space point. Formulation of a quantum field theory in continuous space-time requires the introduction of a gauge fixing term and a corresponding Faddeev-Popov ghost. The essence of local gauge invariance describing the classical theory is inherited by the quantum theory in the form of the global (BRST) invariance identified by Becchi, Rouet and Stora, [17], and Tyutin, [18]. Kugo and Ojima were able to devise a condition for colour confinement in a continuous quantum filed theory by showing that physical, gauge invariant (BRST singlet), particles are also colour singlet, [19]. This condition stems from using a well defined BRST-exact expression to define the colour charge Q^a, and, in the Landau gauge, it is possible to explore the Kugo-Ojima criterion in terms of the Faddeev-Popov ghost 2-point function. The propagator of the Faddeev-Popov ghost is not fundamental but has a dipole behaviour at low momenta. This feature, referred to as ghost enhancement, is not present explicitly when perturbative results obtained using the Faddeev-Popov gauge fixed QCD Lagrangian are extrapolated to the infrared sector. Gribov pointed out that in Yang-Mills theory, the covariant gauge condition typically employed by Faddeev and Popov to fix the gauge in the non-Abelian Lagrangian contains an ambiguity. Whilst Faddeev and Popov succeeded in correctly implementing a choice of gauge into the Yang-Mills Lagrangian, a further issue remained in the fact that expressions used to isolate this choice were inadequate for a non-Abelian gauge theory. The ambiguity results from the occurrence of zeros in the Faddeev-Popov operator, either side of which it is possible for different gauge configurations to satisfy the same naive gauge fixing condition. In a local region, the neighbourhood of the origin of configuration space, where perturbation theory is valid, there is no such ambiguity and perturbative calculations using the Faddeev-Popov gauge fixed Lagrangian are adequate to describe ultraviolet behaviour. However, to fix the gauge properly, globally, the issue of Gribov copies must be addressed when constructing the path integral for the theory. Gribov succeeded in doing this by restricting the domain of integration in the path integral to the region of configuration space contained within the first Gribov horizon, [20]. This is defined by the region, containing the origin, in which the Faddeev-Popov operator is strictly positive. Consequently, the domain of integration in the path integral is restricted, and a natural mass parameter, γ, called the Gribov mass emerges. The Gribov mass is not an independent parameter of Yang-Mills theory, it is nonperturbative and satisfies a gap equation. The gap equation derives from restriction of the domain of integration in the path integral to the region contained within the horizon Ω, using the no pole condition. The no pole condition refers specifically to the Faddeev-Popov (FP) ghost propagator, where of course, the FP ghost is the device used to promote the determinant of the FP operator into the exponent. The expression no pole stems from the requirement that this propagator should have no singularity (pole) other than at vanishing momentum. When the no pole condition, restriction to the region contained within the first Gribov horizon, is implemented on the one loop correction to the gluon propagator, it turns out to display exactly the behaviour described by the Kugo-Ojima confinement criteria, a dipole behaviour at vanishing momentum commonly referred to as enhancement.

Following the work of Gribov, a method for restricting the domain of functional integration to the region contained within the first Gribov horizon suitable for use in a more general treatment of gauge theories was developed by Zwanziger, [21]. The original work considered in the first part of this thesis derives from using this solution, which has come to be known as the Gribov-Zwanziger Lagrangian, to calculate the two loop gap equation incorporating an arbitrary quark mass in the $\overline{\text{MS}}$ renormalization scheme and investigate the implications for Faddeev-Popov ghost enhancement. As we shall see, the form of the *standard* gluon propagator which emerges from implementing the restriction to the Gribov region includes a complex width. If we are to perform meaningful calculations using the Gribov-Zwanziger Lagrangian, it is important that the complex nature of the gluon propagator, used during the integration process, is not still present in the final answer. Fortuitously, this actually proves to be a useful, additional, safety check on the correctness of the calculations. The original work considered here builds on existing loop studies, the new feature of the formal calculation we do is that it includes a real mass scale and a complex width. As such, dealing with the complex nature of the gluon propagator presents special difficulties that were not present in previous loop calculations carried out using this model.

The appearance of a complex width in the Gribov type gluon propagator described above, leads naturally to the question of mass in a non-Abelian gauge theory. In particular we consider that instead of the usual method for generating spontaneous symmetry breaking by introducing an elementary scalar field, symmetry breaking is in fact the result of a dynamically generated condensate of two vector particles. Indeed, pioneering work in methods of spontaneous symmetry breaking, [22], stressed that its origins might well be dynamical. The difficulties associated with building a realistic field theory with dynamical symmetry breaking are primarily concerned with the issue of preserving renormalizability. An additional complication arises when we are considering a theory with asymptotic freedom, such as QCD. Since the primary reason QCD is the leading candidate for accurately describing the strong interaction is that it has the property of asymptotic freedom, incorporating a dynamical symmetry breaking mechanism which destroyed this property would result in a massive theory that is no longer fit for purpose.

The second part of the thesis looks at possible scenarios for the dynamical generation of a gluon mass and how they can be incorporated into the QCD Lagrangian using a composite operator with general form $(A^a_\mu)^2$. The motivation for this work is related to a formalism developed for use with a two-dimensional fermion field theory with asymptotic freedom, the Gross-Neveu model, [23]. In particular, if a physically meaningful $(A^a_\mu)^2$ type operator is to be incorporated into QCD this is most likely to be achieved using the local composite operator (LCO) formalism used to derive renormalizability and identify a non-zero vacuum expectation value for the $\bar{\psi}\psi$ condensate considered in the Gross-Neveu model. The original work presented in the second part of the thesis is concerned with identifying the three loop anomalous dimension for a particular, gauge invariant mass operator candidate, $F^a_{\mu\nu}[(D^2)^{-1}]^{ab}F^b_{\mu\nu}$. It is hoped that using the LCO formalism, it will be possible to derive a two loop effective action for this operator, where due to the additional complications associated with renormalization in the presence of a composite field, the three loop anomalous dimension will be required to calculate a two loop vacuum expectation value.

Chapter 2

Gribov Copies and Gribov Horizons

2.1 Copies and horizons

Since the work of Gribov, [20], it has been well known that the freedom to perform gauge transformations leads to ambiguities in the path integral formulation of a non-Abelian quantum field theory. Faddeev and Popov realized a perturbative interpretation of non-Abelian quantum field theory, [2]. Restricting the generating functional to a consideration of fields that satisfy a particular linear covariant gauge fixing condition, it is possible to use the path integral formalism to produce Green's functions that give meaningful results in the ultraviolet sector. In effect what Faddeev and Popov achieved was a systematic procedure for selecting a single representative vector potential A^a_μ satisfying a particular gauge condition, say $\partial_\mu A^a_\mu = 0$, the Landau gauge, such that at short distances there are no potentials $\tilde{A}^a_\mu$ related to A^a_μ by a gauge transformation,

$$A_\mu \to \tilde{A}_\mu = u^\dagger \partial_\mu u + u^\dagger A_\mu u = A_\mu + u^\dagger(\partial_\mu u + [A_\mu, u]) \quad , \tag{2.1}$$

with the same divergence. The gauge transformation, (2.1), is defined using the connection, A_μ, related to the vector potential through the relationship,

$$A_\mu(x) \equiv A^a_\mu(x)\tau^a \quad , \tag{2.2}$$

where, τ^a, is the generator of some Lie algebra,

$$\left[\tau^a, \tau^b\right] = i f^{abc}\tau^c \quad . \tag{2.3}$$

Also, u is an element of the gauge group such that, infinitesimally, $u = e^{i\omega^b\tau^b}$. The Faddeev-Popov quantization formula for Euclidean Yang-Mills theory, in the Landau gauge, is given by

$$\mathcal{Z} = \mathcal{N} \int \mathcal{D}A_\mu \delta(\partial A) \left[\det\left(\mathcal{M}^{ab}(A)\right)\right] e^{-\frac{1}{4g^2}\int d^4x F^a_{\mu\nu}F^a_{\mu\nu}} \ . \tag{2.4}$$

Using the measure (2.4), it is possible to fix the ultraviolet sector of Yang-Mills theory in the Landau gauge so that only physically in-equivalent gauge fields are considered. Here, and in what follows, $\mathcal{N}$ is an overall normalization factor. The Yang-Mills coupling is denoted by g, the invariant action is defined using the Yang-Mills field strength tensor $F^a_{\mu\nu}$ and the factor $\mathcal{M}^{ab}(A)$ is the Faddeev-Popov operator where, in the conventions of [24], adopted for the purposes of highlighting the ambiguity,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu \ , \tag{2.5}$$

and

$$\mathcal{M}^{ab}(A) = -\partial_\mu \left(D^{ab}_\mu\right) = -\partial_\mu \left(\partial_\mu \delta^{ab} - f^{abc} A^c_\mu\right) \ . \tag{2.6}$$

This convention was chosen so that the Faddeev-Popov operator $\mathcal{M}^{ab}(A)$, given in terms of a vector potential A^c_μ, can be represented using a colour singlet operator $\mathcal{M}(A)$, given in terms of the connection A_μ in the commutator expression,

$$\mathcal{M}(A)u \quad = \quad -\partial_\mu(\partial_\mu u + [A_\mu, u]) \tag{2.7}$$

Defining the Faddeev-Popov operator in this way is particularly useful when introducing the concept of a Gribov copy.

In general, it is not true that two gauge fields A_μ and $\tilde{A}_\mu$ related by a gauge transformation (2.1) cannot satisfy the same gauge condition $\partial_\mu A_\mu = \partial_\mu \tilde{A}_\mu = f$, where f may be equal to 0 or represent a different constraint. The remaining part of this section reviews a detailed consideration of this statement given in [24]. In the Landau gauge, the statement that two physically equivalent fields can satisfy an identical gauge fixing constraint is given by the expression,

$$\partial_\mu(u^\dagger(\partial_\mu u + [A_\mu, u])) = 0 \ , \tag{2.8}$$

where, recalling that $\partial_\mu A_\mu = 0$, we see that any solutions to the equation

$$\partial_\mu u^\dagger \partial_\mu u + u^\dagger \partial_\mu \partial_\mu u + \partial_\mu u^\dagger A_\mu u + u^\dagger A_\mu \partial_\mu u = 0 \ , \tag{2.9}$$

destroy the Landau gauge fixing procedure of Faddeev and Popov. By considering infinitesimal gauge transformations, $u = 1 + \sigma$, where $\sigma = i\omega^a \tau^a$, and $\sigma \ll 1$, the equation for Landau gauge pathology, (2.9), is given by,

$$\partial^2 \sigma - (\partial_\mu \sigma) A_\mu + A_\mu (\partial_\mu \sigma) = \partial_\mu(\partial_\mu \sigma + [A_\mu, \sigma]) = 0 \ . \tag{2.10}$$

By interpreting the colour singlet Faddeev-Popov operator

$$\mathcal{M} = -\partial_\mu(\partial_\mu \cdot + [A_\mu, \cdot]) \ , \tag{2.11}$$

as a kind of Schrödinger equation where the connection, A_μ, plays the role of a potential and $\det(\mathcal{M})$ enters into the Faddeev-Popov quantization formula, (2.4), Gribov demonstrated that the impossibility of a globally correct gauge fixing procedure for a non-Abelian quantum field theory was closely related to the possibility of generating *Gribov* copies. The simplest example of a condition for the occurrence of a Gribov copy is given by an infinitesimal gauge transformation on a *large* valued potential A_μ producing a zero energy solution, $\epsilon = 0$, in the Schrödinger equation interpretation of the Faddeev-Popov operator,

$$-\partial_\mu(\partial_\mu\psi + [A_\mu, \psi]) = \epsilon(A)\psi \ . \tag{2.12}$$

For *small* values of the potential A_μ, (2.12) is solvable for positive values of ϵ only. More precisely, denoting by $\epsilon_1(A), \epsilon_2(A), \epsilon_3(A), \cdots$, the eigenvalues corresponding to a given field configuration A_μ, one has that, for small A_μ, all $\epsilon_i(A)$ are positive. This is precisely the situation for the ultraviolet sector of Quantum Chrodynamics (QCD) where, because of a property known as asymptotic freedom, [12],[13], Faddeev and Popov were able to produce an accurate formulation of a perturbative non-Abelian quantum field theory. For a sufficiently large value of the potential A_μ, outside of the ultraviolet sector of QCD where the constituent fields are not asymptotically free, we obtain a vanishing eigenvalue, $\epsilon_{0,1}(A) = 0$. As the field increases further the Schrödinger equation interpretation produces negative energy solutions, bound states of the Faddeev-Popov operator. If we continue along a gauge orbit of A_μ in function space via a gauge transformation (2.1) we observe a second vanishing eigenvalue $\epsilon_{0,2}(A)$. Using the Schrödinger equation interpretation Gribov divided the functional space of the fields A_μ into regions $C_0, C_1, C_2, \cdots, C_n$ supporting n bound states, negative eigenvalues $\epsilon_{-1,n}(A)$, of the Faddeev-Popov operator, illustrated schematically in Fig(2.1). The regions are separated by lines $l_1, l_2, l_3, \cdots, l_n$ on which the Faddeev-Popov operator has zero energy solutions. Inside the region C_0, the operator only supports positive eigenvalues, and the line l_1 enclosing this region is called the first Gribov horizon.

In order to proceed, it is necessary to demonstrate the existence of a normalizable zero energy mode of the Faddeev-Popov operator, say χ, for a field A_i which lies on a boundary/horizon l_n and obeys the Coulomb gauge condition $\partial_i A_i = 0$. This is achieved in a concrete manner using the gauge group SU(2) in three dimensions, [25], [26]. Secure in the knowledge that there exists a normalizable zero mode, χ, of the Faddeev-Popov operator for a field A_i satisfying the Coulomb gauge condition on the boundary l_1 we assume that there does exist a solution valid in four dimensions and return to a consideration of Euclidean Yang-Mills theory in four space-time dimensions. Using a zero mode of the Faddeev-Popov operator it is possible to demonstrate how an infinitesimal gauge transformation on a field A_μ, inside the region C_0 and close to the boundary/horizon l_1, can lead to the possibility of producing a Gribov copy $\tilde{A}_\mu$, inside

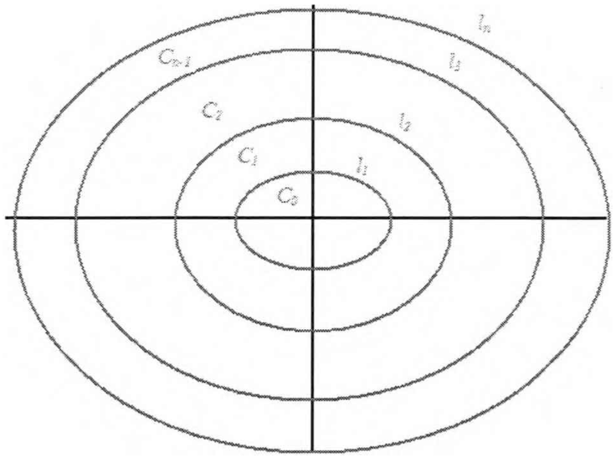


Figure 2.1: The Gribov Horizons

the region C_1 also close to the horizon l_1, that satisfies a gauge condition identical to A_μ. The initial field A_μ has two components,

$$A_\mu = C_\mu + a_\mu \ , \tag{2.13}$$

where C_μ is a transverse field, on the Gribov horizon l_1, such that there exists a normalizeable zero mode of the Faddeev-Popov operator given by,

$$\partial_\mu(\partial_\mu \phi_0 + [C_\mu, \phi_0]) = 0 \ , \tag{2.14}$$

and a_μ is a perturbation. By definition, the field A_μ belongs to the region C_0. Using the standard perturbation theory of quantum mechanics, it is possible to show that, if A_μ, close to l_1, is located in C_0, $\epsilon(a) > 0$, there is an equivalent field, $\tilde{A}_\mu = A_\mu + D_\mu(C)\phi_0$, close to l_1, which is located in C_1, with eigenvalue $\epsilon(\tilde{a}) = -\epsilon(a) < 0$. A detailed consideration of this statement, [24], is included in Appendix A. Also, that derivation can be generalized to fields close to any horizon l_n.

2.2 Elimination of copies by restriction to the first horizon

The analysis of the previous section suggests that one way to eliminate the existence of Gribov copies is to restrict the domain of integration in the path integral to a consideration of the region C_0 only, where the Faddeev-Popov determinant is positive definite. We represent this by adding a further restriction to the partition function $\mathcal{Z}$,

$$\begin{aligned} \mathcal{Z} &= \mathcal{N} \int \mathcal{D}A_\mu \mathcal{D}c \mathcal{D}\bar{c} \delta(\partial A) e^{-(S_{YM} + \int d^4x \bar{c}^a \partial_\mu D_\mu^{ab} c^b)} \mathcal{V}(C_0) \\ &= \mathcal{N} \int \mathcal{D}A_\mu \delta(\partial A) e^{-S_{YM}} \det(-\partial_\mu(\partial_\mu \delta^{ab} - f^{abc} A_\mu^c)) \mathcal{V}(C_0) \ , \end{aligned} \tag{2.15}$$

where the new factor $\mathcal{V}(C_0)$ symbolizes the idea of a restriction in the domain of *functional* integration to the region C_0. The possibility of a meaningful interpretation of this factor is investigated by considering a result for the connected two-point ghost function $\langle\bar{c}^a(x)c^b(y)\rangle$.

$$\begin{aligned}\langle\bar{c}^a(x)c^b(y)\rangle &= \mathcal{N}\int \mathcal{D}A_\mu \mathcal{D}c\mathcal{D}\bar{c}\delta(\partial A)\bar{c}^a(x)c^b(y)e^{-(S_{YM}+\int \mathrm{d}^4x\bar{c}^a\partial_\mu D^{ab}_\mu c^b)}\mathcal{V}(C_0)\\ &= \mathcal{N}\int \mathcal{D}A_\mu\delta(\partial A)e^{-S_{YM}}\det(-\partial_\mu D_\mu)\left[(\partial_\mu D_\mu)^{-1}\right]^{ab}_{xy}\mathcal{V}(C_0)\ , \quad (2.16)\end{aligned}$$

When $\mathcal{V}(C_0) = 1$ the perturbative connected two-point ghost function is given by,

$$\begin{aligned}\langle\bar{c}^a(x)c^b(y)\rangle &= \delta^{ab}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathcal{G}(k)e^{ik(x-y)}\ ,\\ \mathcal{G}(k) &= \frac{1}{k^2}\frac{1}{\left(1-\frac{11g^2C_A}{48\pi^2}\ln\left(\frac{\Lambda^2}{k^2}\right)\right)^{9/44}}\ , \quad (2.17)\end{aligned}$$

where Λ is an ultraviolet cut off. We can see from the expression for $\mathcal{G}(k)$ in (2.17) that the perturbative result for the connected ghost two-point function contains two singularities, k_1 and k_2, located at, $k_1^2 = 0$ and $k_2^2 = \Lambda^2\exp(-48\pi^2/11g^2C_A)$. For $k^2 < k_2^2$, $\mathcal{G}(k)$ becomes complex indicating that the Faddeev-Popov operator has ceased to be a positive definite quantity and that we have left the region C_0 in function space. The presence of the factor $\mathcal{V}(C_0)$ in the partition function is intended to restrict the domain of the functional measure by requiring that the connected ghost two-point function has no singularities for non vanishing k. That is, including the factor $\mathcal{V}(C_0)$ in the functional measure removes the singularity at k_2.

To recap, the region C_0 is defined as the set of all transverse connections A_μ for which the Faddeev-Popov operator does not allow zero modes,

$$C_0 \in \{A_\mu,\ \partial A = 0,\ -\partial_\mu\left(\partial_\mu\cdot + [A_\mu,\cdot]\right) > 0\}\ . \quad (2.18)$$

As such, within this region the Faddeev-Popov operator remains a positively defined quantity and is invertible, an essential property that can be seen from the second expression for the connected ghost two-point function (2.17). To proceed with a characterization of the factor $\mathcal{V}(C_0)$, $\mathcal{G}(k;A)$ is denoted by the colour singlet Fourier transform of $[-\partial_\mu(\partial_\mu - f^{abc}A^a_\mu)]^{-1}$,

$$\mathcal{G}(k;A) = \sum_{ab}\frac{\delta^{ab}}{N_A^2-1}\left\langle k\left|\left[-\partial_\mu\left(\partial_\mu\delta^{ab} - f^{abc}A^c_\mu\right)\right]^{-1}\right|k\right\rangle\ , \quad (2.19)$$

where, for the colour index a,

$$1 \le a \le N_A\ , \quad (2.20)$$

with the requirement that $\mathcal{G}(k;A)$ has no poles for non vanishing momenta k. The expression for the connected, colour singlet, ghost two-point function is given by

$$\sum_{ab}\frac{\delta^{ab}\langle\bar{c}^a(x)c^a(y)\rangle}{N_A^2-1} = \mathcal{N}\int\mathcal{D}A_\mu\mathcal{D}\bar{c}\mathcal{D}c\delta(\partial A)\frac{\bar{c}^a(x)c^a(y)}{N_A^2-1}e^{-(S_{YM}+\int \mathrm{d}^4x\bar{c}^a\partial_\mu D^{ab}_\mu c^b)}$$

$$= \mathcal{N} \int \mathcal{D}A_\mu \delta(\partial A) e^{-S_{YM}} \mathcal{G}(x, y; A) \ , \tag{2.21}$$

where the gauge field A^a_μ is a classical external field. Using Wick's theorem it is possible to evaluate $\langle \bar{c}^a(x) c^b(y) \rangle$ to second order in perturbation theory, [24]. The full detail of the calculation is included in Appendix A, where it is shown that

$$\mathcal{G}(k; A) \approx \frac{1}{k^2} \frac{1}{(1 - \sigma(k, A))} \ , \tag{2.22}$$

with

$$\sigma(k, A) = \frac{N_A}{N_A^2 - 1} \frac{1}{k^2} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{(k-q)_\mu k_\nu}{(q-k)^2} A^a_\mu(-q) A^a_\nu(q) \ . \tag{2.23}$$

2.3 The no pole condition

Still following the analysis of [24], it is possible to establish the no pole condition for the 2-point ghost function. From (2.22) it follows that the no pole at non vanishing momentum k can be simply stated as

$$\sigma(k, A) < 1 \ . \tag{2.24}$$

Further simplification is possible by observing that, in the Landau gauge,

$$q_\mu A^a_\mu(q) = 0 \ . \tag{2.25}$$

From

$$q_\mu A^a_\mu(-q) A^a_\nu(q) = q_\nu A^a_\mu(-q) A^a_\nu(q) = 0 \ , \tag{2.26}$$

we can set

$$A^a_\mu(-q) A^a_\nu(q) = \omega(A) \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \ . \tag{2.27}$$

Contracting both sides with $\delta_{\mu\nu}$, in four dimensions, gives

$$\omega(A) = \frac{1}{3} A^a_\lambda(q) A^a_\lambda(-q) \ . \tag{2.28}$$

Putting this simplification into the pole regulator gives,

$$\sigma(k, A) = \frac{1}{3} \frac{N_A}{N_A^2 - 1} \frac{k_\mu k_\nu}{k^2} \mathcal{I}_{\mu\nu}(k) \ , \tag{2.29}$$

where,

$$\mathcal{I}_{\mu\nu}(k) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(k-q)^2} (A^a_\lambda(q) A^a_\lambda(-q)) \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \ . \tag{2.30}$$

Now, as we will see shortly, the quantity $\langle A^a_\lambda(q) A^a_\lambda(-q) \rangle$ decreases with q^2, so that $\sigma(k, A)$ decreases as k^2 increases. Hence, as the no pole condition it is possible to use

$$\sigma(0, A) < 1 \ , \tag{2.31}$$

where

$$\sigma(0,A) = \frac{N_A}{N_A^2-1}\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{q^2}\langle A_\lambda^a(q)A_\lambda^a(-q)\rangle \ . \tag{2.32}$$

The last expression follows by setting

$$\mathcal{I}_{\mu\nu}(0) \;=\; \int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{q^2}\langle A_\lambda^a(q)A_\lambda^a(-q)\rangle\left(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right) \;=\; \mathcal{J}\delta_{\mu\nu} \ , \tag{2.33}$$

and contracting both sides with $\delta_{\mu\nu}$, in four dimensions, leading to

$$\mathcal{J} \;=\; \frac{3}{4}\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{q^2}\langle A_\lambda^a(q)A_\lambda^a(-q)\rangle \ . \tag{2.34}$$

Putting this into the expression for $\sigma(0,A)$,

$$\begin{aligned}\sigma(0,A) &= \frac{1}{3}\frac{N_A}{N_A^2-1}\frac{k_\mu k_\nu}{k^2}\mathcal{I}_{\mu\nu} \\ &= \frac{1}{3}\frac{N_A}{N_A^2-1}\mathcal{J} \\ &= \frac{1}{4}\frac{N_A}{N_A^2-1}\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{q^2}\langle A_\lambda^a(q)A_\lambda^a(-q)\rangle \ . \end{aligned} \tag{2.35}$$

2.4 An expression for $\mathcal{V}(C_0)$

According to [20], the expression for the factor $\mathcal{V}(C_0)$ which implements the no pole condition (2.22) in the path integral may be taken as

$$\mathcal{V}(C_0) = \theta(1-\sigma(0,A)) \ , \tag{2.36}$$

where $\theta(x)$ is the step function

$$\theta(x) = \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x<0 \end{cases} \ .$$

For the partition function $\mathcal{Z}$ we have

$$\mathcal{Z} = \mathcal{N}\int \mathcal{D}A_\mu \delta(\partial A)e^{-S_{YM}}\det(-\partial_\mu D_\mu)\theta(1-\sigma(0,A)) \ . \tag{2.37}$$

Using the integral representation for the step function

$$\theta(x) = \int_{-\infty+\epsilon}^{i\infty+\epsilon}\frac{\mathrm{d}\beta}{2\pi i\beta}e^{\beta x} \ , \tag{2.38}$$

leads to a useful expression

$$\mathcal{Z} = \mathcal{N}\int \frac{\mathrm{d}\beta}{2\pi i\beta}\mathcal{D}A_\mu\delta(\partial A)e^{\beta(1-\sigma(0,A))}e^{-S_{YM}}\det(-\partial_\mu D_\mu) \ , \tag{2.39}$$

which is suitable for analyzing the gluon propagator. This is considered in the next section.

2.5 The gluon propagator in the Landau gauge

In order to calculate the propagator, it is necessary to retain only the quadratic terms in (2.39), that is,

$$\begin{aligned} \mathcal{Z}_{quad} &= \mathcal{N}\int \frac{\mathrm{d}\beta}{2\pi i\beta}\mathcal{D}A_\mu e^{\beta(1-\sigma(0,A))} \\ &\quad\times \exp\left[-\frac{1}{4g^2}\int \mathrm{d}^4x(\partial_\mu A_\nu^a-\partial_\nu A_\mu^a)^2-\frac{1}{2g^2\alpha}\int \mathrm{d}^4x(\partial_\mu A_\mu^a)^2\right] \end{aligned} \tag{2.40}$$

where it is understood that the limit $\alpha\to 0$ must be taken at the end in order to recover the Landau gauge where $\sigma(0,A)$ is defined. Moving to momentum space we get

$$\begin{aligned} \mathcal{Z}_{quad} &= \mathcal{N}\int \frac{\mathrm{d}\beta}{2\pi i\beta}\mathcal{D}A\exp\left[-\frac{1}{2g^2}\sum_q A_\mu^a(q)\left(q^2\delta_{\mu\nu}+\left(\frac{1}{\alpha}-1\right)q_\mu q_\nu\right)A_\nu^a(-q)\right] \\ &\quad\times\exp(\beta)\exp\left[-\beta\left(\frac{N_A}{N_A^2-1}\right)\frac{1}{4}\frac{1}{V}\sum_q\frac{1}{q^2}\left(A_\lambda^a(q)A_\lambda^a(-q)\right)\right] \\ &= \mathcal{N}\int \frac{\mathrm{d}\beta e^\beta}{2\pi i\beta}\mathcal{D}A\exp\left[-\frac{1}{2g^2}\sum_q A_\mu^a(q)\mathcal{Q}_{\mu\nu}^{ab}A_\nu^b(-q)\right] \quad , \end{aligned} \tag{2.41}$$

where

$$\mathcal{Q}_{\mu\nu}^{ab}=\left(q^2\delta_{\mu\nu}+\left(\frac{1}{\alpha}-1\right)q_\mu q_\nu+\frac{\beta N_A g^2}{N_A^2-1}\frac{1}{2}\frac{1}{V}\frac{1}{q^2}\delta_{\mu\nu}\right)\delta^{ab} \quad . \tag{2.42}$$

Applying the standard formula

$$\mathcal{Z}_{quad}=\mathcal{N}\int\frac{\mathrm{d}\beta e^\beta}{2\pi i\beta}\left(\det\mathcal{Q}_{\mu\nu}^{ab}\right)^{-1/2} \quad . \tag{2.43}$$

Gribov was not successful in producing a rigorous characterization of the functional space $\mathcal{V}(C_0)$. As such, some of the manipulations in the following lines may appear crude. From

$$\begin{aligned} \left(\det\mathcal{Q}_{\mu\nu}^{ab}\right)^{-1/2} &= \exp\left[-\frac{1}{2}\ln\left(\det\mathcal{Q}_{\mu\nu}^{ab}\right)\right] \\ &= \exp\left[-\frac{3}{2}\frac{N_A^2}{N_A^2-1}\sum_q\ln\left(q^2+\frac{\beta N_A g^2}{N_A^2-1}\frac{1}{2V}\frac{1}{q^2}\right)\right] \quad , \end{aligned} \tag{2.44}$$

we obtain,

$$\mathcal{Z}_{quad}=\mathcal{N}\int\frac{\mathrm{d}\beta}{2\pi i}e^{f(\beta)} \quad , \tag{2.45}$$

with,

$$f(\beta)=\beta-\ln(\beta)-\frac{3}{2}(N_A^2-1)\sum_q\ln\left(q^2+\frac{\beta N_A g^2}{N_A^2-1}\frac{1}{2V}1q^2\right) \quad . \tag{2.46}$$

Expression (2.45) can be evaluated at the saddle point

$$\mathcal{Z}_{quad}\approx e^{f(\beta_0)} \quad , \tag{2.47}$$

where β_0 is determined by the minimum condition

$$f'(\beta_0) = 0 \tag{2.48}$$

so that

$$1 - \frac{1}{\beta_0} - \frac{3}{4}\frac{N_A g^2}{V}\sum_q \frac{1}{q^4 + \frac{\beta_0 N_A g^2}{N_A^2 - 1}\frac{1}{2V}} = 0 \ . \tag{2.49}$$

Taking the thermodynamic limit, that is when the volume V tends to infinity,

$$\gamma^4 = \frac{\beta_0 N_A g^2}{N_A^2 - 1}\frac{1}{2V}\Big|_{V\to\infty} \ , \tag{2.50}$$

gives the following gap equation for the parameter γ

$$\frac{3N_A g^2}{4}\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^4+\gamma^4} = 1 \ , \tag{2.51}$$

where the term $1/\beta_0$ has been neglected in the thermodynamic limit. We can see from (2.51) that the parameter γ is specifically defined by an expression which includes the Yang-Mills coupling constant. The parameter γ has the dimension of mass and using

$$\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^4+\gamma^4} = \frac{\Omega_4}{(2\pi)^4}\int_0^{\Lambda}\frac{dq q^3}{q^4+\gamma^4} = \frac{1}{16\pi^2}\ln\left(\frac{\Lambda^2}{\gamma^2}\right) \ , \tag{2.52}$$

where $\Omega_4 = 2\pi^2$ is the four-dimensional solid angle and Λ is the ultraviolet cutoff, finally

$$\gamma^2 = \Lambda^2 \exp\left[-\frac{64\pi^2}{3N_A g^2}\right] \ . \tag{2.53}$$

To obtain the gluon propagator we return to the expression (2.41) which is now evaluated at the saddle point value β_0

$$\mathcal{Z}_{quad} = \mathcal{N}'\int \frac{d\beta e^{\beta}}{2\pi i\beta}\mathcal{D}A \exp\left[-\frac{V}{2g^2}\int \frac{d^4q}{(2\pi)^4}A_\mu^a(q)\mathcal{Q}_{\mu\nu}^{ab}A_\nu^b(-q)\right] \ , \tag{2.54}$$

with

$$\mathcal{Q}_{\mu\nu}^{ab} = \left(\left(q^2 + \frac{\gamma^4}{q^2}\right)\delta_{\mu\nu} + \left(\frac{1}{\alpha} - 1\right)q_\mu q_\nu\right)\delta^{ab} \ . \tag{2.55}$$

The Landau gauge gluon propagator is obtained in the usual manner by evaluating the inverse of $\mathcal{Q}_{\mu\nu}^{ab}$ and taking the limit $\alpha \to 0$. A straightforward calculation gives

$$\mathcal{D}_{\mu\nu}^{ab}(q) = \langle A_\mu^a(q)A_\nu^b(-q)\rangle = \delta^{ab}\frac{q^2}{q^4+\gamma^4}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \ . \tag{2.56}$$

For $q^2 \gg \gamma^2$, we recover the usual perturbative behaviour

$$\mathcal{D}_{\mu\nu}^{ab}(q)\Big|_{\text{(ultraviolet limit)}} = \delta^{ab}\frac{1}{q^2}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \ . \tag{2.57}$$

For $q^2 < \gamma^2$, the infrared region, the Landau gauge gluon propagator is strongly suppressed.

2.6 The ghost propagator in the Landau gauge

For the ghost propagator we have, in terms of the no pole condition,

$$\mathcal{G}(k;A) \approx \frac{1}{k^2}\frac{1}{1-\sigma(k,A)} , \tag{2.58}$$

where, in the Landau gauge,

$$\begin{aligned}\sigma(k,A) &= \frac{N_A}{N_A^2-1}\frac{1}{k^2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{(k-q)_\mu k_\nu}{(k-q)^2}\langle A_\mu^a(-q)A_\nu^a(q)\rangle \\ &= \frac{N_A k_\mu k_\nu}{k^2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{(k-q)^2}\frac{q^2}{q^4+\gamma^4}\left(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)\end{aligned} \tag{2.59}$$

The infrared behavior, $k \approx 0$, of $(1-\sigma(k))$ is examined by making use of the gap equation (2.51) and of

$$\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^4+\gamma^4}\left(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right) = \frac{3}{4}\delta_{\mu\nu}\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^4+\gamma^4} , \tag{2.60}$$

It follows that

$$N_A g^2\frac{k_\mu k_\nu}{k^2}\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^4+\gamma^4}\left(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right) = 1 . \tag{2.61}$$

Thus, for $(1-\sigma(k))$, we obtain

$$\begin{aligned}(1-\sigma(k)) &= N_A g^2\frac{k_\mu k_\nu}{k^2}\int\frac{d^4q}{(2\pi)^4}\left(1-\frac{q^2}{(k-q)^2}\right)\frac{1}{q^4+\gamma^4}\left(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right) \\ &= N_A g^2\frac{k_\mu k_\nu}{k^2}\int\frac{d^4q}{(2\pi)^4}\frac{(k^2-2kq)}{(k-q)^2}\frac{1}{q^4+\gamma^4}\left(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right) \\ &\equiv N_A g^2\frac{k_\mu k_\nu}{k^2}\mathcal{P}_{\mu\nu}(k) ,\end{aligned} \tag{2.62}$$

where

$$\mathcal{P}_{\mu\nu}(k) = \int\frac{d^4q}{(2\pi)^4}\frac{(k^2-2kq)}{(k-q)^2}\frac{1}{q^4+\gamma^4}\left(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right) . \tag{2.63}$$

From this expression one sees that $\mathcal{P}_{\mu\nu}(k)$ is convergent and non singular at $k=0$. In fact

$$\mathcal{P}_{\mu\nu}(0) = 0 , \tag{2.64}$$

from which it follows that, for $k \approx 0$,

$$\mathcal{P}_{\mu\nu}(k)_{k\to 0} \approx k^2\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2}\frac{1}{q^4+\gamma^4}\left(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right) . \tag{2.65}$$

Since

$$\begin{aligned}\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2}\frac{1}{q^4+\gamma^4}\left(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right) &= \frac{3}{4}\delta_{\mu\nu}\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2}\frac{1}{q^4+\gamma^4} \\ &= \delta_{\mu\nu}\frac{3}{4}\frac{\Omega_4}{(2\pi)^4}\int dq\frac{q}{q^4+\gamma^4} \\ &= \delta_{\mu\nu}\frac{3}{8}\frac{\Omega_4}{(2\pi)^4}\frac{\pi}{2}\frac{1}{\gamma^2} \\ &= \delta_{\mu\nu}\frac{3}{128\pi}\frac{1}{\gamma^2} ,\end{aligned} \tag{2.66}$$

which gives

$$\mathcal{P}_{\mu\nu}(k)_{k\to 0} \approx k^2 \delta_{\mu\nu} \frac{3}{128\pi} \frac{1}{\gamma^2} \,. \tag{2.67}$$

Therefore

$$(1 - \sigma(k))_{k\to 0} \approx \frac{3N_A g^2}{128\pi} \frac{1}{\gamma^2} k^2 \,, \tag{2.68}$$

and, for the infrared behavior of the ghost propagator

$$\mathcal{G}(k)_{k\to 0} \approx \frac{128\pi\gamma^2}{3N_A g^2} \frac{1}{k^4} \,. \tag{2.69}$$

One sees thus that, while the gauge propagator is suppressed in the infrared, the ghost propagator is enhanced at $k \approx 0$, being indeed more singular than $1/k^2$.

2.7 Gribov problem

In this chapter, we have shown that fixing the divergence of the potential in non-Abelian gauge theories does not, globally, fix its gauge. The task of improving the Faddeev-Popov quantization procedure is reduced to imposing the additional limitation in the functional measure that the integration range in the space of non-Abelian fields is restricted to those for which the eigenvalues of the Faddeev-Popov determinant are strictly positive. The additional constraint is not significant in the ultraviolet sector, the property of asymptotic freedom is not affected. In order to simplify the mathematics, all analysis is done in four dimensional Euclidean space-time, this makes it more difficult to understand the physical content. Despite this limitation, the general statement, (2.15), that integration should only be performed over physically in-equivalent fields holds, regardless of the nature of the space. For Yang-Mills theory, we observe a gluon propagator suppressed in the infrared region, and a ghost propagator which is enhanced. An enhanced ghost propagator is consistent with the Kugo-Ojima confinement criterion for colour charged particles to decouple from the physical spectrum, [19]. The analysis of the preceding pages is taken from a comprehensive review, [24], describing, faithfully and in more detail, the calculations of Gribov. That is, the work described in this chapter is a detailed study of Gribov's original formulation of a Gribov copy, and subsequent calculations leading to modified infrared propagators for the gluon and ghost fields. In the next chapter, we will consider the significant contribution made by Zwanziger and then use it to do calculations in following chapters. Implementing a restriction to the region contained within the first Gribov horizon of more practical use, it is necessary to consider/formulate the Gribov problem in a very different way, as we will see. However, it should be understood that the aims of the next chapter are closely related to what is described here.

Chapter 3

The Gribov-Zwanziger Lagrangian

3.1 Practical interpretation of the Gribov horizon

Following the work of Gribov, [20], Zwanziger embarked upon the task of identifying a practical interpretation of the Gribov region C_0, described in terms of two criteria. These are, the Landau gauge condition

$$\partial_\mu A_\mu = 0 \ , \tag{3.1}$$

and positivity of the Faddeev-Popov operator

$$\mathcal{M}(A) = -\left(\partial^2 - A_\mu \partial_\mu\right) \geq 0 \ . \tag{3.2}$$

Also, since the work of Gribov, the region contained within the first Gribov horizon, the principal region C_0 has come to be known as Ω, and the boundary of the horizon itself is often referred to as $\partial\Omega$. As such, the conditions (3.1) and (3.2) may be combined to define a succinct definition of the principal Gribov region using,

$$\Omega = \{A_\mu | \partial_\mu A_\mu = 0 \text{ and } -\partial_\mu D_\mu(A) > 0\} \ . \tag{3.3}$$

In analogy with the original formulation, the work of Zwanziger necessarily involves two parts. The first is to identify some important properties that describe, or are satisfied by, the region Ω. The second is to use these properties to restrict consideration of gauge theories to the region contained within $\partial\Omega$. Unlike the original work of Gribov, it is not contained in a single paper, reflecting the broader scope and practical importance of this formulation.

Beginning with a description of Ω, three initial properties were established [27],

1. Ω is convex

2. Every gauge orbit intersects Ω

3. Ω is bounded in all directions.

The properties are derived by making reference to a fixed background potential $\bar{A}$ satisfying the condition that

$$D_\mu^{ab}(\bar{A})\omega^b = \partial_\mu \omega^a - f^{abc}\bar{A}_\mu^b \omega^c = 0 \quad , \tag{3.4}$$

where, D is the covariant derivative and ω represents an infinitesimal gauge transformation, has no solutions $\omega \neq 0$. Using $\bar{A}$, the strong gauge constraint (3.3), for the region $\Omega_{\bar{A}}$ is defined such that

$$\begin{aligned} D(\bar{A})(A-\bar{A}) &= 0 \\ -D(\bar{A})D(A) &> 0 \ \forall\, \omega \ . \end{aligned} \tag{3.5}$$

The variable $a = A - \bar{A}$ represents a quantum fluctuation on a classical background. We now review each property separately.

1. The fact that the region $\Omega_{\bar{A}}$ is convex, is proved by defining the operator

$$H(a) \equiv -D(\bar{A})D(A) = -D^2(\bar{A}) - aD(\bar{A}) \quad , \tag{3.6}$$

linear in a, such that

$$\langle \omega^\dagger H(a)\omega \rangle = \alpha \langle \omega^\dagger H(a_1)\omega \rangle + (1-\alpha)\langle \omega^\dagger H(a_2)\omega \rangle \tag{3.7}$$

where

$$\begin{aligned} a &= \alpha a_1 + (1-\alpha)a_2 \\ 0 &\leq \alpha \leq 1 \ , \end{aligned} \tag{3.8}$$

from (3.5), the right hand side of (3.7) is positive for all ω. If a_1 and a_2 are in $\Omega_{\bar{A}}$, a is also in $\Omega_{\bar{A}}$; this region is convex. The interior of $\Omega_{\bar{A}}$ consists of those points a in a hyperplane $D(\bar{A})a = 0$ such that $H(a)$ is a positive definite operator.

2. That every gauge orbit intersects the region $\Omega_{\bar{A}}$ is determined by making reference to the generalized chiral action

$$\begin{aligned} S_{A\bar{A}}(u) &= -\frac{1}{N}\int \mathrm{d}^4x \mathrm{tr}\left(A_\mu^u - \bar{A}_\mu\right)^2 \\ &= -\frac{1}{N}\int \mathrm{d}^4x \mathrm{tr}\left(u^\dagger A_\mu u + \frac{i}{g}u^\dagger \partial_\mu u - \bar{A}_\mu\right)^2 \ . \end{aligned} \tag{3.9}$$

If u is replaced by ue^ω, where ω is an element of the Lie algebra, then to second order in ω

$$\begin{aligned} S_{A\bar{A}}(ue^\omega) &= S_{A\bar{A}}(u) - 2\int \mathrm{d}^4x \omega^a D^{ab}(\bar{A})(A^u - A)^b \\ &+ \int \mathrm{d}^4x D_\mu^{ab}(A^u)\omega^b D_\mu^{ac}(\bar{A})\omega^c + O(\omega^3) \ . \end{aligned} \tag{3.10}$$

The points along the orbit A^u through A at which the action is stationary are precisely the points of intersection of the orbit A^u with the hyperplane $D(\bar{A})(A^u - \bar{A}) = 0$. Also, at a minimum of $S_{A\bar{A}}(u)$, the Hermitian quadratic form $-D(\bar{A})D(A^u)$ is positive, so that every minimum of $S_{A\bar{A}}(u)$ occurs when A^u is in $\Omega_{\bar{A}}$. As such, every gauge orbit must intersect $\Omega_{\bar{A}}$.

3. Finally, too establish that the region $\Omega_{\bar{A}}$ is bounded in every direction, consider a non zero fluctuation a satisfying, $D(\bar{A})a = 0$. For some positive μ_0, $\mu_0 a$ is on the boundary $\partial\Omega_{\bar{A}}$, and μa lies inside or outside of the region according to, [28],

$$\begin{aligned} 0 \;\leq\; \mu \;<\; \mu_0 \\ \mu \;>\; \mu_0 \;. \end{aligned} \tag{3.11}$$

The eigenvalue $\lambda_i(\mu a)$ of

$$H(\mu a) = -D^2(\bar{A}) - \mu a D(\bar{A}) \quad , \tag{3.12}$$

are continuous in μ, and as we have seen, for $\mu = 0$ the least eigenvalue $\lambda_0(0)$ is positive. Hence, for sufficiently small μ, $\lambda_0(\mu a)$ is strictly positive and μa is in $\Omega_{\bar{A}}$. Since this region is convex, demonstrating that for sufficiently large positive μ, μa lies outside $\Omega_{\bar{A}}$ will show the existence of a boundary in all directions. For this purpose, it is necessary to construct a gauge transformation $\omega^a(x)$ such that

$$\langle \omega^\dagger H(\mu a)\omega \rangle < 0 \quad , \tag{3.13}$$

for sufficiently large μ. Choosing a point x_0 such that $a^b_\nu(x_0) \neq 0$ defined in a region $\mathcal{O}$ small enough that at least one component of $a^b_\nu(x)$ does not change sign in this neighbourhood. Now, let $\phi(x)$ be a smooth function in $\mathcal{O}$, such that

$$\int \mathrm{d}^4x |\phi(x)|^2 = 1 \tag{3.14}$$

These requirements are sufficient to ensure that

$$b^c_\nu = \int \mathrm{d}^4x \; \phi^\dagger(x) a^c_\nu(x)\phi(x) \neq 0 \; . \tag{3.15}$$

Now, defining a unit vector $\hat{k}_\nu$, such that $n^c = \hat{k}_\nu b^c_\nu \neq 0$, we choose

$$\omega^a(x) = e^{-kx}\phi(x)u^a \quad , \tag{3.16}$$

where, $k = c\hat{k}$ and c is a free parameter, and u^a is a constant normalized colour vector, $u^\dagger u = 1$. The expectation value $\langle \omega^\dagger(-aD(\bar{A}))\omega \rangle$ is given by

$$M(a) = icu^{\dagger a} f^{abc} n^b u^c - \langle (\phi u)^\dagger a D(\bar{A})\phi u \rangle \quad , \tag{3.17}$$

where the matrix $(\Sigma \cdot n)^{ac} = if^{abc}n^c$ is Hermitian and has at least one non-vanishing eigenvalue λ. Choosing u to be the corresponding eigenvector

$$M(a) = \lambda c - \langle (\phi u)^\dagger a D(\bar{A})\phi u \rangle \; . \tag{3.18}$$

For c sufficiently large and opposite in sign to λ (3.17) is seen to be negative. Moreover, because $M(a)$ is linear in a,

$$\langle \omega^\dagger H(\mu a)\omega\rangle = -\langle \omega^\dagger(-D^2(\bar{A}))\omega\rangle + \mu M(a) \quad , \tag{3.19}$$

and so, for sufficiently large positive μ

$$\langle \omega^\dagger H(\mu a)\omega\rangle < 0 \quad , \tag{3.20}$$

establishing a boundary for $\Omega_{\bar{A}}$ in all directions.

3.2 Infinite dimensional ellipsoid

It is possible to impose definite boundaries on the region Ω, by considering a gauge field $A^c_\mu(x)$ on a Euclidean base manifold which is a periodic box of edge L, parametrized using coordinates x_μ which vary in the interval $0 \leq x_\mu \leq L$. The gauge field $A^c_\mu(x)$ is expanded in a Fourier series, [29]:

$$A^c_\mu(x) \;=\; \frac{1}{V^{1/2}}\sum_k a^c_\mu(k)\exp(ikx) \quad , \tag{3.21}$$

where inside the box,

$$k_\mu \;=\; \frac{2\pi n_\mu}{L} \quad , \tag{3.22}$$

so that n_μ runs over all integers, and

$$k_\mu a^c_\mu(k) \;=\; 0 \quad , \tag{3.23}$$

transversality is assumed throughout. The gluon propagator is expressed in terms of the Fourier coefficients by

$$G^{bc}_{\mu\nu}(x) \;=\; \frac{1}{V}\sum_k g^{bc}_{\mu\nu}(k)\exp(ikx) \quad , \tag{3.24}$$

where

$$g^{bc}_{\mu\nu}(k) \;=\; \langle a^b_\mu(k)a^c_\nu(-k)\rangle \quad . \tag{3.25}$$

As in [20] a convenient definition of (3.2) is that the lowest eigenvalue $\lambda_0(a)$, regarded as a function of the Fourier coefficients $a^c_\mu(k)$, be positive:

$$\lambda_0(a) \geq 0 \quad . \tag{3.26}$$

Applying this constraint, it is possible to explore some simple bounds on the region Ω using degenerate perturbation theory through the definition

$$\mathcal{M}(A) = \mathcal{M}_0 + \mathcal{M}_1(A) = -\partial^2 + A_\mu\partial_\mu \quad , \tag{3.27}$$

such that

$$\mathcal{M}u = \mathcal{M}_0 u + \mathcal{M}_1 u = \lambda(a)u \quad , \tag{3.28}$$

where,

$$\begin{aligned}
(\mathcal{M}_0 u)(k) &= k^2 u(k) \\
(\mathcal{M}_1 u)(k) &= -\frac{i}{V^{1/2}} \sum_q a_\mu(k-q) q_\mu u(q) \quad .
\end{aligned} \tag{3.29}$$

Using (3.29), it is possible to derive a matrix M, with eigenvalues $\lambda_{n,2}(a)$, given to second order in perturbation theory, in terms of the Fourier components $a^c_\mu(k)$. Instead of calculating the boundary of the region Ω_2 directly it is instructive to derive a transparently simple bound on it in terms the sum of the eigenvalues of M, given by

$$\mathrm{tr}M = 2dC_A \left(\frac{2\pi}{L}\right)^2 \left(1 - \sum_{k,n,b} \frac{C_A |a^b_n(k)|^2}{dN_A k^2 V}\right) \quad , \tag{3.30}$$

where d is the space-time dimension, and C_A is the quadratic Casimir operator in the adjoint representation and N_A is the dimension of the adjoint representation. Since Ω_2 is defined by the condition that the lowest eigenvalue of $M(a)$ must be positive, all points a in Ω_2 also satisfy the weaker condition that $\mathrm{tr}M(a)$ be positive. It follows that all points a in Ω_2 satisfy

$$\sum_{k,n,b} \frac{C_A}{dN_A} \frac{a^b_{Rn}(k)^2 + a^b_{In}(k)^2}{k^2 V} \leq 1 \quad , \tag{3.31}$$

where,

$$a^b_n(k) = a^b_{Rn}(k) + i a^b_{In}(k) \quad . \tag{3.32}$$

Exploring this weaker condition has made it possible to show that Ω_2 is contained within an infinite dimensional real ellipsoid, E, with principal axes that are aligned along the real and imaginary parts of the $a^b_n(k)$, with semi-major axes,

$$R(k) = \left(\frac{dN_A k^2 V}{C_A}\right)^{1/2} \quad . \tag{3.33}$$

Importantly, the ellipsoid E is bounded in all directions and this bound is proportional to $|k|$. It is reasonable to expect that a bound on $a(k)$ satisfying a stronger condition will scale with $|k|$ in a similar or stronger manner such that the low momenta components of the gauge field A will be strongly suppressed as a consequence.

Using the weaker condition has made it possible to derive a transparently simple upper bound on the region Ω, defined in terms of an infinite dimensional ellipsoid

$$E = \sum_{k,n,b} \frac{|a^b_n(k)|^2}{ck^2 V} \quad . \tag{3.34}$$

This simple interpretation, derived in terms of the eigenvalues of the Faddeev-Popov operator $\mathcal{M}(A)$ to second order in perturbation theory, implies that low momenta components of the gauge field A are suppressed by a restriction to the region contained within the Gribov horizon.

3.3 A Gribov region suitable for use in gauge theories

Having developed a simple interpretation in terms of the ellipsoid E, it is necessary to develop a device for implementing this restriction in terms of a functional measure appropriate for use with gauge theories, [30]. Using perturbation theory again, it is stipulated that positivity of the Faddeev-Popov operator $\mathcal{M}(A)$ can be reduced to positivity of the $2dN_A \times 2dN_A$ matrix

$$\kappa = k_0^2 P_0 + \kappa_1 + \kappa_R \quad , \tag{3.35}$$

where the perturbation is performed around the first non-trivial eigenspace of

$$\mathcal{M}_0 = -\partial^2 \quad , \tag{3.36}$$

belonging to the eigenvalue $\lambda_0 = (2\pi/L)^2$. The corresponding eigenvectors are labeled $|k_0, b\rangle$, where b is a colour index, and the components of the matrix κ, (3.35), are given by,

$$\begin{aligned} k_0^2 &= \left(\frac{2\pi}{L}\right)^2 \\ P_0 &= \sum_{k_0,b} |k_0, b\rangle\langle k_0, b| \\ \kappa_1 &= \frac{-ik_{0\mu}}{V} \int \mathrm{d}^d x f^{abc} A_\mu^c(x) \\ \kappa_R &= \frac{k_0^2}{V} \int \mathrm{d}^d x \mathrm{d}^d y A_\mu(x) \mathcal{M}^{-1}(x, y; A) A_\nu(y) \quad , \end{aligned} \tag{3.37}$$

where the term κ_R results from the whole perturbation series being summed according to

$$\kappa_R = \sum_{n=2}^{\infty} \kappa_n \quad . \tag{3.38}$$

As before, it is necessary to replace the condition that the least eigenvalue of the matrix $\kappa(A)$ be positive by the weaker condition that $\mathrm{tr}\kappa(A)$ is positive and the obligatory warning: The correct Gribov region may be smaller than the one we find here, in which case the impact on the physics derived using this formalism will be less drastic than that implied by a restriction to the correct Gribov region. The weaker condition leads to an explicit expression with a simple volume dependence, [30],

$$\mathrm{tr}\kappa(A) = k_0^2 2dN_A - k_0^2 4C_A Q[A] \quad . \tag{3.39}$$

In his original derivation, Zwanziger chooses to normalize the functional $Q[A]$ with a factor C_A^{-1}, such that

$$Q[A] \;=\; -\frac{1}{2C_A V}\int \mathrm{d}^d x \mathrm{d}^d y \mathrm{tr}\left[A_\mu(x)\mathcal{M}^{-1}(x,y;A)A_\nu(y)\right] \ . \tag{3.40}$$

It is now possible to turn to a consideration of the correct measure for a non-Abelian gauge theory incorporating the factor $\theta(\lambda[A])$ which restricts the domain of integration to the region contained within the Gribov horizon. Replacing $\theta(\lambda[A])$ by $\theta(\mathrm{tr}\kappa[A])$, the correct measure is given by, [30],

$$\mathrm{d}\mu_c \equiv \mathcal{D}A\delta(\partial_\mu A_\mu)\exp\left(-\frac{1}{g^2}S_{cl}\right)\det(\mathcal{M}[A])\theta\left(c-\frac{1}{g^2}Q[A]\right) \ , \tag{3.41}$$

where

$$c=\frac{dN_A}{2C_A g^2} \ . \tag{3.42}$$

In the lowest order $\mathcal{M} = \mathcal{M}_0 = -\partial^2$, and the shape of the Gribov horizon coincides precisely with the infinite dimensional ellipsoid E that was considered in a first approximation.

In order to proceed, it is necessary to examine the properties of the infinite dimensional ellipsoid, E (3.34), more closely using a simplified model, where

$$E_0 = \frac{1}{V}\sum_k |a(k)|^2 p(k^2) = c \ . \tag{3.43}$$

Now, making a change of variable from $a(k)$ to

$$y_k \equiv a(k)p^{1/2}(k^2) \ . \tag{3.44}$$

means that the ellipsoid $E_0 = c$ in A-space is mapped into the sphere in y-space given by

$$y^2 \equiv \sum_k y_k^2 = Vc \ . \tag{3.45}$$

An integral over y extends over the volume of the ball in y-space which is bounded at the radius $r \equiv (y^2)^{1/2} = R \equiv (Vc)^{1/2}$. Now let the functional integral be regularized by a cut-off in momentum space, so that the dimension of A-space or y-space is the finite but large number N. In a Euclidean space of dimension N, the volume element in the radial variable $r \equiv (y^2)^{1/2}$ is $r^{N-1}\mathrm{d}r$, and as N grows without limit, the volume of the ball of radius R, becomes concentrated at its surface, the sphere of radius R. Thus, in the lowest order, it is correct to replace $\theta(c - g^{-2}Q[A])$ by $\delta(c - g^{-2}Q[A])$, leading to a consideration of the measure,

$$\mathrm{d}\mu_c \equiv \mathrm{d}A\delta(\partial_\mu A_\mu)\exp\left(-\frac{1}{g^2}S_{cl}\right)\det(\mathcal{M}[A])\delta\left(c-\frac{1}{g^2}Q[A]\right) \ . \tag{3.46}$$

Higher order effects are considered as perturbations of this lowest order shape.

In analogy with, and in the language of, statistical mechanics, we proceed by assuming equivalence of the micro-canonical and canonical ensembles for this measure, so that the δ-function may in turn be replaced by the corresponding Boltzmann factor,

$$d\mu_c \equiv \mathcal{D}A\delta(\partial_\mu A_\mu)\exp\left[-\frac{1}{g^2}\left(S_{cl}+\gamma S_1[A]\right)\right]\det(\mathcal{M}[A]) \quad . \tag{3.47}$$

Here $S_1 \equiv VQ[A]$ is given by

$$S_1 \quad = \quad -\frac{1}{2C_A}\int d^dx d^dy \text{tr}\left[A_\mu(x)\mathcal{M}^{-1}(x,y;A)A_\nu(y)\right] \quad . \tag{3.48}$$

Using (3.42), the value of the thermodynamic parameter γ is determined by the condition

$$\frac{dN_A}{2C_A g^2} = c(\gamma) \equiv \frac{1}{g^2}\langle Q[A]\rangle \quad , \tag{3.49}$$

where the expectation value refers to the Faddeev-Popov measure and (3.49) is analogous to the Gribov gap equation, [20], (2.51). Here also, the thermodynamic parameter γ and the coupling constant g are clearly related, [30]. By translational invariance, $c(\gamma)$ may be written

$$c(\gamma) \quad = \quad -\frac{1}{2C_A g^2}\int d^dx \text{tr}\langle A_\mu(x)\mathcal{M}^{-1}(x,0;A)A_\nu(0)\rangle \quad . \tag{3.50}$$

This condition provides an absolute normalization for the gauge field A. It expresses the fact that the measure is supported on the Gribov horizon.

To see the significance of the new non-local term S_1 in the action (3.47), consider a point A inside the Gribov region Ω where the eigenvalues of $\mathcal{M}[A]$ are all positive. As A approaches the boundary of Ω, the lowest eigenvalue $\lambda[A]$ of $\mathcal{M}[A]$ approaches zero. Because $\mathcal{M}^{-1}$ appears in S_1, the probability is suppressed by a factor

$$\propto \exp\left(-\frac{\text{const.}}{\lambda[A]}\right) \quad . \tag{3.51}$$

The new term dominates the dynamics in the infrared region. In particular, it strongly suppresses the gluon propagator in the infrared so that the tree level propagator vanishes like k^2 as $k \to 0$.

The action implementing the restriction to the Gribov region is cast into a local form using a generalization of the well known formulae

$$\int dy dy^* \exp(-y^* A y) \quad = \quad \det A \tag{3.52}$$

$$\int dy dy^* \exp(-y^* A y + \eta^* y + y^* \eta) \quad = \quad \det A \exp(\eta^* A^{-1}\eta) \quad . \tag{3.53}$$

The localized action is given by, [30],

$$\begin{aligned}\exp\left(-\frac{\gamma S_1}{g^2}\right) \quad &= \quad (\det\mathcal{M})^{dN_A/2}\int\mathcal{D}\phi\mathcal{D}\bar{\phi}\exp\left[-\frac{1}{g^2}\int d^dx\right. \\ &\quad \times\left.\left(\frac{1}{2}\bar{\phi}^{ab}_\mu\mathcal{M}^{bc}\phi^{ac}_\mu + \frac{\gamma^2}{\sqrt{C_A}}f^{abc}(\phi^{ac}_\mu A^b_\mu - \bar{\phi}^{ac}_\mu A^b_\mu\right)\right] \quad .\end{aligned} \tag{3.54}$$

In this local action, $\phi^{ab}_\mu(x)$ and $\bar{\phi}^{ab}_\mu(x)$ are complex Bose fields. In analogy with the Faddeev-Popov prescription, the determinant is incorporated into the exponent by introducing additional, anti-commuting, ghost fields

$$(\det\mathcal{M})^{dN_A/2} = \int \mathcal{D}\omega\mathcal{D}\bar{\omega}\exp\left(-\frac{1}{g^2}\int d^dx\bar{\omega}^{ab}\mathcal{M}^{bc}\omega^{ac}\right) \ . \tag{3.55}$$

The complete measure, expressing the restriction of Yang-Mills theory fixed in the Landau gauge to the region contained within the first Gribov horizon and using local field parameters, [31],[21], is given by

$$d\mu_\gamma = \mathcal{D}A\mathcal{D}c\mathcal{D}\bar{c}\mathcal{D}\phi\mathcal{D}\bar{\phi}\mathcal{D}\omega\mathcal{D}\bar{\omega}\exp\left(-\frac{1}{g^2}S_\gamma\right) \ , \tag{3.56}$$

where

$$\begin{aligned} S_\gamma &= S_{YM} + S_0 + \int d^4x\left[\frac{\gamma^2}{\sqrt{2}}(f^{abc}A^a_\mu\phi^{bc}_\mu - f^{abc}A^a_\mu\bar{\phi}^{bc}_\mu) - \frac{dN_A\gamma^4}{2g^2}\right] \\ &= \int d^4x\left[\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \alpha b^a b^a + b^a\partial_\mu A^a_\mu + \bar{c}^a\partial_\mu D_\mu c^a + \bar{\phi}^{ab}_\mu\partial_\nu(D_\nu\phi_\mu)^{ab}\right. \\ &\qquad - \bar{\omega}^{ab}_\mu\partial_\nu(D_\nu\omega_\mu)^{ab} - gf^{abc}\partial_\nu\bar{\omega}^{ae}_\mu(D_\nu c)^b\phi^{ec}_\mu \\ &\qquad \left. + \gamma^2(f^{abc}A^a_\mu\phi^{bc}_\mu - f^{abc}A^a_\mu\bar{\phi}^{bc}_\mu)\right] \ . \end{aligned} \tag{3.57}$$

The local classical action (3.57) describes a Yang-Mills theory, fixed in the Landau gauge using the Faddeev-Popov method, subject to the additional constraint that all eigenvalues of the Faddeev-Popov operator are greater than or equal to zero, that is

$$[\mathcal{M}(A)]^{ab} = -\partial_\mu D^{ab}_\mu \geq 0 \ . \tag{3.58}$$

The auxiliary field b^a, the Lagrange multiplier, is used to derive the linear covariant gauge fixing. It is included in the action for use in the BRST analysis of renormalizability in the Landau gauge where α is set equal to zero. In the next chapter, the term $\propto \alpha$ will be reintroduced for the purposes of deriving propagators.

In addition to the usual parameters of Yang-Mills theory the action (3.57) includes the the additional localizing fields $\{\phi^{ab}_\mu, \bar{\phi}^{ab}_\mu\}$ and $\{\omega^{ab}_\mu, \bar{\omega}^{ab}_\mu\}$, each pair are spin-1 and carry two colour indices in which each pair is antisymmetric. The former are commuting Bose fields whilst the latter describe anti-commuting Grassmann variables. The new Bose fields are introduced to implement the Gribov restriction in a local way, (3.54), and the new Grassmann quantities are necessary to accommodate the determinant generated by this process. The Gribov parameter γ, which has the dimension of mass, may be regarded as a statistical mechanical parameter, and is defined using the non-local gap equation

$$\left\langle A^a_\mu(x)\frac{1}{\partial_\nu D_\nu}A^a_\mu(x)\right\rangle = \frac{dN_A}{C_A g^2} \ . \tag{3.59}$$

The formula (3.59) defines the horizon condition. It is analogous to Gribov's no-pole condition, [20], which was derived using the Faddeev-Popov ghost two-point function and inserted into the expression for the gluon 2-point function to express the restriction of the functional measure to the region contained within the first Gribov horizon.

3.4 Renormalization

We have shown above, that it is possible to express the restriction of a gauge theory to the region contained within the Gribov horizon using the local action S_γ. This immediately raises the question of renormalizability: does a quantum theory, derived using the classical action (3.57), contain any new divergences or anomalies not present in Yang-Mills theory, and if so, are they renormalizable? Considering that part of the action S_γ which implements the horizon function

$$S_{\text{horizon}} = \gamma^2(f^{abc}A^a_\mu\phi^{bc}_\mu - f^{abc}A^a_\mu\bar{\phi}^{bc}_\mu) \ , \tag{3.60}$$

and recalling that γ has the dimension of mass, the action (3.57) may be regarded as a massive field theory, where the mass term is expressed using the composite fields that are mixed in $\{A^a_\mu, \phi^{ab}_\mu, \bar{\phi}^{ab}_\mu\}$. Following the procedure of [32], by introducing variable sources for the composite fields it is possible to express the correlation functions of a massive theory in terms of the correlation functions of a massless theory, where $f^{abc}A^a_\mu\phi^{bc}_\mu$ and $f^{abc}A^a_\mu\bar{\phi}^{bc}_\mu$ describe mass insertions. Turning attention to the massless theory

$$\begin{aligned} S_\gamma|_{\gamma=0} &= S_{YM} + S_0 \\ &= \int d^4x \left[\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + b^a\partial_\mu A^a_\mu + \bar{c}^a\partial_\mu D_\mu c^a + \bar{\phi}^{ab}_\mu\partial_\nu(D_\nu\phi_\mu)^{ab}\right. \\ &\qquad \left. - \bar{\omega}^{ab}_\mu\partial_\nu(D_\nu\omega_\mu)^{ab} - gf^{abc}\partial_\nu\bar{\omega}^{ae}_\mu(D_\nu c)^b\phi^{ec}_\mu\right] \ , \end{aligned} \tag{3.61}$$

except for the last term, the Faddeev-Popov ghost increasing vertex, the new parameters in the action (3.61) are given using the expressions generated by a localization procedure suitable for use with gauge theories. The extra term is defined by a translation of the variable ω,

$$\omega' \to \omega + \mathcal{M}^{-1}g\partial Dc\phi \ , \tag{3.62}$$

keeping $\bar{\omega}$ fixed. The presence of this additional term means that the massless theory, fixed in the Landau gauge, is left invariant by the following nilpotent BRST transformations

$$\begin{aligned} sA^a_\mu &= -(D_\mu c)^a & sc^a &= \frac{1}{2}gf^{abc}c^bc^c \\ s\bar{c}^a &= b^a & sb^a &= 0 \\ s\phi^a_i &= \omega^a_i & s\omega^a_i &= 0 \\ s\bar{\omega}^a_i &= \bar{\phi}^a_i & s\bar{\phi}^a_i &= 0 \ , \end{aligned}$$

$$sS_\gamma|_{\gamma=0} = 0 \ . \tag{3.63}$$

The horizon function (3.60), or mass insertion, violates the BRST invariance of the local massless action (3.61). It is introduced into the BRST conserving theory using suitably chosen local sources where the original action S_γ is recovered when the sources attain their physical values.

In addition to BRST invariance, the Landau gauge fixed massless action (3.61) displays a global $U(f)$ symmetry, $f = 4(N_A^2 - 1)$, with respect to the composite index $i = (\mu, b) = 1, \cdots, f$, of the additional fields $\{\phi_\mu^{ab}, \bar{\phi}_\mu^{ab}, \omega_\mu^{ab}, \bar{\omega}_\mu^{ab}\}$. By setting

$$\{\phi_\mu^{ab}, \bar{\phi}_\mu^{ab}, \omega_\mu^{ab}, \bar{\omega}_\mu^{ab}\} = \{\phi_i^a, \bar{\phi}_i^a, \omega_i^a, \bar{\omega}_i^a\} \ , \tag{3.64}$$

the massless action now reads,

$$\begin{aligned} S_\gamma|_{\gamma=0} &= \int \mathrm{d}^4x \left[\frac{1}{4}F_{\mu\nu}^a F_{\mu\nu}^a + b^a \partial_\mu A_\mu^a + \bar{c}^a \partial_\mu D_\mu c^a + \bar{\phi}_i^a \partial_\nu (D_\nu \phi_i)^a \right. \\ &\qquad \left. - \bar{\omega}_i^a \partial_\nu (D_\nu \omega_i)^a - g f^{abc} \partial_\nu \bar{\omega}_i^a (D_\nu c)^b \phi_i^c \right] \ . \end{aligned} \tag{3.65}$$

The $U(f)$ invariance is expressed using the Ward identity

$$U_{ij} S_\gamma|_{\gamma=0} = 0 \ , \tag{3.66}$$

where

$$U_{ij} = \int \mathrm{d}^4x \left(\phi_i^a \frac{\delta}{\delta\phi_j^a} - \bar{\phi}_j^a \frac{\delta}{\delta\bar{\phi}_i^a} + \omega_i^a \frac{\delta}{\delta\omega_j^a} - \bar{\omega}_j^a \frac{\delta}{\delta\bar{\omega}_i^a}\right) \ . \tag{3.67}$$

In order to prove renormalizability for the massless action, it is useful to define an additional quantum number Q_f for the auxiliary fields using the diagonal operator, $Q_f = U_{ii}$.

Proving renormalizability for a theory with mass insertions derives from using the technique of local sources, [32]. Following this, in [21], the composite fields are coupled to external sources, $\{U_\mu^{ai}, V_\mu^{ai}, M_\mu^{ai}, N_\mu^{ai}\}$, in the complete action

$$\begin{aligned} S_\gamma &= \int \mathrm{d}^4x \left[\frac{1}{4}F_{\mu\nu}^a F_{\mu\nu}^a + b^a \partial_\mu A_\mu^a + \bar{c}^a \partial_\mu D_\mu c^a + \bar{\phi}_i^a \partial_\mu (D_\mu \phi_i)^a \right. \\ &\qquad - \bar{\omega}_i^a \partial_\mu (D_\mu \omega_i)^a - g f^{abc} \partial_\mu \bar{\omega}_i^a (D_\mu c)^b \phi_i^c \\ &\qquad + M_\mu^{ai} (D_\mu \phi_i)^a - U_\mu^{ai} (D_\mu \omega_i)^a + g f^{abc} U_\mu^{ai} (D_\mu c)^b \phi_i^c \\ &\qquad + N_\mu^{ai} (D_\mu \omega_i) a + V_\mu^{ai} (D_\mu \bar{\phi}_i)^a - g f^{abc} V_\mu^{ai} (D_\mu c)^b \bar{\omega}_i^c \\ &\qquad \left. + M_\mu^{ai} V_\mu^{ai} - U_\mu^{ai} N_\mu^{ai} \right] \ . \end{aligned} \tag{3.68}$$

The sources have BRST transformations,

$$\begin{aligned} sU_\mu^{ai} &= M_\mu^{ai} & sM_\mu^{ai} &= 0 \\ sV_\mu^{ai} &= N_\mu^{ai} & sN_\mu^{ai} &= 0 \ , \end{aligned} \tag{3.69}$$

given this, it is possible to express the complete action, (3.68), now denoted by Σ, using

$$\begin{aligned} \Sigma &= \int \mathrm{d}^4x \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a \\ &\quad + s \int \mathrm{d}^4x \left(-\bar{c}^a \partial_\mu D_\mu c^a + \bar{\omega}_i^a \partial_\mu (D_\mu \phi_i)^a + U_\mu^{ai} (D_\mu \phi_i)^a + V_\mu^{ai} (D_\mu \bar{\omega}_i)^a + U_\mu^{ai} V_\mu^{ai}\right) \\ &\quad + \int \mathrm{d}^4x \left(-K_\mu^a (D_\mu c)^a + \frac{1}{2} g f^{abc} L^a c^b c^c\right) \ , \end{aligned} \tag{3.70}$$

where in the last line, as is customary [32], non-linear BRST transformations in (3.63) are also coupled to external sources K^a_μ and L^a. Due to the nilpotency of the BRST transform, $s^2 = 0$, it is straightforward to see that the complete action Σ is BRST invariant. When the sources attain their physical values,

$$\begin{aligned} N^{ab}_{\mu\nu} &= U^{ab}_{\mu\nu} = 0 \\ M^{ab}_{\mu\nu} &= -V^{ab}_{\mu\nu} = \gamma^2 \delta_{\mu\nu}\delta^{ab} \ . \end{aligned} \tag{3.71}$$

Quantum numbers for all the fields and sources, Faddeev-Popov ghost number and Q_f charge, together with the dimension are detailed in Table 3.1.

	A^a_μ	c^a	$\bar{c}^a$	b^a	ϕ^a_i	$\bar{\phi}^a_i$	ω^a_i	$\bar{\omega}^a_i$	K	L	M	N	U	V
dimension	1	0	2	2	1	1	1	1	4	2	2	2	2	2
ghost number	0	1	-1	0	0	0	1	-1	-1	-2	0	1	-1	0
Q_f charge	0	0	0	0	1	-1	1	-1	0	0	-1	1	-1	0

Table 3.1: Quantum numbers for Gribov-Zwanziger parameters.

The values in Table 3.1 play an important role in the renormalization criteria for the model. The classical action Σ, (3.70), constructed in a manifestly BRST invariant way, displays a rich algebraic structure, [21],[34]. In particular, the following symmetries are important for proving renormalizability, [34].

1. The Slavnov-Taylor identity

$$\mathcal{S}(\Sigma) = 0 \ , \tag{3.72}$$

where

$$\begin{aligned} \mathcal{S}(\Sigma) = \int d^4x \Bigg(&\frac{\delta\Sigma}{\delta K^a_\mu}\frac{\delta\Sigma}{\delta A^a_\mu} + \frac{\delta\Sigma}{\delta L^a}\frac{\delta\Sigma}{\delta c^a} + b^a\frac{\delta\Sigma}{\delta\bar{c}^a} + \omega^a_i\frac{\delta\Sigma}{\delta\phi^a_i} \\ &+ \bar{\phi}^a_i\frac{\delta\Sigma}{\delta\bar{\omega}^a_i} + M^{ai}_\mu\frac{\delta\Sigma}{\delta U^{ai}_\mu} + N^{ai}_\mu\frac{\delta\Sigma}{\delta V^{ai}_\mu}\Bigg) \ , \end{aligned} \tag{3.73}$$

and the algebraic renormalization process will utilize the linearized version of this identity

$$\begin{aligned} \mathcal{B}_\Sigma = \int d^4x \Bigg(&\frac{\delta\Sigma}{\delta K^a_\mu}\frac{\delta}{\delta A^a_\mu} + \frac{\delta\Sigma}{\delta A^a_\mu}\frac{\delta}{\delta K^a_\mu} + \frac{\delta\Sigma}{\delta L^a}\frac{\delta}{\delta c^a} + \frac{\delta\Sigma}{\delta c^a}\frac{\delta}{\delta L^a} \\ &+ b^a\frac{\delta}{\delta\bar{c}^a} + \omega^a_i\frac{\delta}{\delta\phi^a_i} + \bar{\phi}^a_i\frac{\delta}{\delta\bar{\omega}^a_i} + M^{ai}_\mu\frac{\delta}{\delta U^{ai}_\mu} + N^{ai}_\mu\frac{\delta}{\delta V^{ai}_\mu}\Bigg) \ . \end{aligned} \tag{3.74}$$

2. In the Landau gauge, the integrated ghost equation of motion [35], leads to the Ward identity,

$$\mathcal{G}^a\Sigma = \Delta^a \ , \tag{3.75}$$

where

$$\mathcal{G}^a = \int d^4x \left(\frac{\delta}{\delta c^a} + gf^{abc} \left[\bar{c}^b \frac{\delta}{\delta b^c} + \phi_i^b \frac{\delta}{\delta \omega_i^c} + \bar{\omega}_i^b \frac{\delta}{\delta \bar{\phi}_i^c} + V_\mu^{bi} \frac{\delta}{\delta N_\nu^{ci}} + U_\mu^{bi} \frac{\delta}{\delta M_\mu^{ci}} \right] \right) , \tag{3.76}$$

and

$$\Delta^a = \int d^4x g f^{abc} \left(K_\mu^b A_\mu^c - L^b c^c \right) . \tag{3.77}$$

3. The Ward identity

$$\mathcal{F}^i \Sigma = 0 , \tag{3.78}$$

where

$$\mathcal{F}^i = \int d^4x \left(c^a \frac{\delta}{\delta \omega_i^a} + \bar{\omega}_i^a \frac{\delta}{\delta \bar{c}^a} + U_\mu^{ai} \frac{\delta}{\delta K_\mu^a} \right) . \tag{3.79}$$

The question of renormalizability is considered by first identifying the most general counterterm Σ_Δ, compatible with the algebraic structure of the model, in particular the identities (3.72), (3.75) and (3.78). Then, renormalizability is proved by showing that the symmetries considered hold to all orders in perturbation theory, that is, they are not anomalous. According to the quantum action principle (QAP), [36],[37],[38], Σ_Δ is given by the most general integrated local functional of dimension four with vanishing ghost and Q_f charge, compatible with the symmetry content of the model, and must also satisfy the identities

$$\frac{\delta \Sigma_\Delta}{\delta b^a} = 0 , \tag{3.80}$$

the (Landau) gauge condition, and

$$\partial_\mu \frac{\delta \Sigma_\Delta}{\delta K_\mu^a} + \frac{\delta \Sigma_\Delta}{\delta \bar{c}^a} = 0 , \tag{3.81}$$

the antighost equation. Using (3.80) and (3.81), it was proven, [21], that Σ_Δ is in fact a functional of only

$$\Sigma_\Delta[A, c, \tilde{K}, L, \tilde{M}, \tilde{N}, \tilde{U}, \tilde{V}] ,$$

where

$$\begin{aligned}
\tilde{K}_\mu^a &= K_\mu^a + \partial_\mu \bar{c}^a + gf^{abc}(U_\mu^{bi} + \partial_\mu \bar{\omega}_i^b)\phi_i^c + gf^{abc} V_\mu^{bi} \bar{\omega}_i^c \\
\tilde{M}_\mu^{ai} &= M_\mu^{ai} + \partial_\mu \bar{\phi}_i^a \\
\tilde{N}_\mu^{ai} &= N_\mu^{ai} + \partial_\mu \omega_i^a \\
\tilde{U}_\mu^{ai} &= U_\mu^{ai} + \partial_\mu \bar{\omega}_i^a \\
\tilde{V}_\mu^{ai} &= V_\mu^{ai} + \partial_\mu \phi_i^a .
\end{aligned} \tag{3.82}$$

The most general allowed counterterm satisfying the linearized Slavnov-Taylor identity $\mathcal{B}_\Sigma$, is found to be, [34],

$$\begin{aligned} \Sigma_\Delta &= a_o \int \mathrm{d}^4x F^a_{\mu\nu} F^a_{\mu\nu} \\ &\quad + \mathcal{B}_\Sigma \int \mathrm{d}^4x \left(a_1 \tilde{K}^a_\mu A^a_\mu + a_2 L^a c^a + a_3 \tilde{U}^{ai}_\mu \tilde{V}^{ai}_\mu \right) \ , \end{aligned} \tag{3.83}$$

where a_0, a_1, a_2 and a_3 are constants. Exploiting the symmetry (3.75) gives

$$a_2 = 0 \ , \tag{3.84}$$

and the symmetry (3.78) implies, [34],

$$a_1 = -a_3 \ . \tag{3.85}$$

Using only the symmetries (3.72), (3.75) and (3.78), it is possible to show that the classical action Σ, (3.70), has two divergences

$$\begin{aligned} \Sigma_\Delta &= a_o \int \mathrm{d}^4x F^a_{\mu\nu} F^a_{\mu\nu} \\ &\quad + a_1 \mathcal{B}_\Sigma \int \mathrm{d}^4x \left(\tilde{K}^a_\mu A^a_\mu - \tilde{U}^{ai}_\mu \tilde{V}^{ai}_\mu \right) \ , \end{aligned} \tag{3.86}$$

which can be absorbed through two independent multiplicative renormalization constants, in complete agreement with ordinary Yang-Mills theory in the Landau gauge [33]. Remarkably, Zwanziger's local action, implementing the restriction of Yang-Mills theory to the region contained within $\partial\Omega$, generates no new divergences or anomalies.

After exploiting the BRST invariance of the massless classical action to establish renormalizability for the model, the issue of specific divergences is tackled using the recursive counterterm construction procedure outlined in [33]. The model has been constructed in such a way that the renormalization properties for the objects of ordinary Yang-Mills theory are not affected. For the additional fields $\{\phi^{ab}_\mu, \bar{\phi}^{ab}_\mu, \omega^{ab}_\mu, \bar{\omega}^{ab}_\mu\}$ the recursive procedure gives the relations, [21],

$$Z_\phi = Z_\omega = Z_c = (Z_A Z_g)^{-1/2} \ , \tag{3.87}$$

where the fields undergo a multiplicative renormalization,

$$\begin{aligned} \phi^{ab}_{\mu r} &= Z^{1/2}_\phi \phi^{ab}_{\mu o} & \bar{\phi}^{ab}_{\mu r} &= Z^{1/2}_\phi \bar{\phi}^{ab}_{\mu o} \\ \omega^{ab}_{\mu r} &= Z^{1/2}_\omega \omega^{ab}_{\mu o} & \bar{\omega}^{ab}_{\mu r} &= Z^{1/2}_\omega \bar{\omega}^{ab}_{\mu o} \ , \end{aligned} \tag{3.88}$$

in which, o and r denote bare and renormalized quantities respectively. Similarly, for the Gribov parameter

$$Z_\gamma = \sqrt{\frac{Z_A}{Z_g}} \ , \tag{3.89}$$

where

$$\gamma_r = Z_\gamma \gamma_o \ . \tag{3.90}$$

In this chapter, we have given a brief overview of the Gribov-Zwanziger action. This began with a review of the properties that describe or are satisfied by the region Ω, given using rather formal reasoning, followed by a perturbative analysis that is more in depth than the original by Gribov. Finally, a *massive* local action is identified and shown to be renormalizable by treating the mass operator as an insertion. Having the correct form of the local renormalizable action, it is now possible to use the full massive theory to perform loop calculations and, in particular, investigate the implications for the gluon propagator beyond tree level and subject the Faddeev-Popov ghost propagator to a more formal study in perturbation theory.

Chapter 4

$\overline{\rm MS}$ scheme renormalization

4.1 The Gribov-Zwanziger Lagrangian and the $\overline{\rm MS}$ scheme

To recap, the improved gauge fixing procedure of Gribov is expressed by Zwanziger in terms of a local renormalizable Lagrangian, here also incorporating quarks to extend the pure Yang-Mills theory to a consideration of QCD, is given by

$$\begin{aligned}\mathcal{L}^{GZ} &= \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A^a_\mu)^2 - \bar{c}^a\partial_\mu D_\mu c^a + i\bar{\psi}^{iI}\not{D}\psi^{iI} \\ &+ \bar{\phi}^{ab}_\mu\partial_\nu(D_\nu\phi_\mu)^{ab} - \bar{\omega}^{ab}_\mu\partial_\nu(D_\nu\omega_\mu)^{ab} - gf^{abc}\partial_\nu\bar{\omega}^{ae}_\mu(D_\nu c)^b\phi^{ec}_\mu \\ &+ \frac{\gamma^2}{\sqrt{2}}(f^{abc}A^a_\mu\phi^{bc}_\mu - f^{abc}A^a_\mu\bar{\phi}^{bc}_\mu) - \frac{dN_A\gamma^4}{4g^2} \quad , \qquad (4.1)\end{aligned}$$

where, in addition to the usual Faddeev-Popov ghosts, $\{c^a, \bar{c}^a\}$, of the canonical gauge fixing procedure, Zwanziger ghosts, $\{\phi^{ab}_\mu, \bar{\phi}^{ab}_\mu\}$ (commuting), and, $\{\omega^{ab}_\mu, \bar{\omega}^{ab}_\mu\}$ (anti-commuting) are also present. The remaining fields are the gluon, A^a_μ, and a massless quark/anti-quark pair $\{\psi^{iI}, \bar{\psi}^{iI}\}$. Indices lie in the ranges,

$$\begin{aligned}1 &\leq a \leq N_A \\ 1 &\leq i \leq N_f \\ 1 &\leq I \leq N_F \quad , \qquad (4.2)\end{aligned}$$

where N_F and N_A are the respective dimensions of the fundamental and adjoint representations of an unspecified semi-simple Lie group and N_f denotes the number of quark flavours. The Yang-Mills field strength tensor $F^a_{\mu\nu}$, is given by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \quad , \qquad (4.3)$$

where, for completeness, we note that g is the Yang-Mills coupling constant. The covariant derivatives are given by

$$D_\mu c^a = \partial_\mu c^a - gf^{abc}A^b_\mu c^c$$

$$
\begin{aligned}
D_\mu \psi^{iI} &= \partial_\mu \psi^{iI} + g\tau^a A^a_\mu \psi^{iI} \\
(D_\mu \phi_\nu)^{ab} &= \partial_\mu \phi^{ab}_\nu - g f^{acd} A^c_\mu \phi^{db}_\nu \ ,
\end{aligned} \tag{4.4}
$$

where, f^{abc} are the structure constants and τ^a are the generators for the gauge group. Although we will only be concerned with the Landau gauge limit, $\alpha = 0$, as with all types of Yang-Mills/QCD Landau gauge studies, the gauge parameter, α, must be included for the purposes of deriving propagators.

The Feynman rules for the vertices may be read off directly from the Lagrangian. The modified behaviour of the tree level gluon propagator stems from the presence of the mixed $A^a_\mu \phi^{bc}_\nu$ terms and the propagators for all fields involved in mixing require special attention. The derivation of the propagators comprising fields in the set $\{A^a_\mu, \phi^{ab}_\mu, \bar{\phi}^{ab}_\mu\}$ is achieved by considering terms of (4.1) quadratic in these fields. Transforming into momentum space that part of the Lagrangian quadratic in members of this set,

$$
\begin{aligned}
\mathcal{L}^{GZ}_{\text{quad}} &= -\frac{1}{2} A^a_\mu(-p) \left[p^2 \eta_{\mu\nu} - \left(1 - \frac{1}{\alpha}\right) p_\mu p_\nu \right] A^a_\nu(p) + \frac{\gamma^2}{\sqrt{2}} f^{abc} A^a_\mu(-p) \phi^{bc}_\mu(p) \\
&\quad - \frac{\gamma^2}{\sqrt{2}} f^{abc} \bar{\phi}^{ab}_\mu(-p) A^c_\nu(p) - p^2 \bar{\phi}^{ab}_\mu(-p) \phi^{ab}_\mu(p) \ ,
\end{aligned} \tag{4.5}
$$

and writing $\mathcal{L}^{GZ}_{\text{quad}}$ in matrix form with respect to the basis $\left\{ \frac{A^a_\mu}{\sqrt{2}}, \phi^{ab}_\mu \right\}$

$$
\mathcal{L}^{GZ}_{\text{quad}} = \frac{1}{2} \left(A^a_\mu(-p), \sqrt{2}\bar{\phi}^{ab}_\mu(-p) \right) \mathbf{X}^{abcd} \begin{pmatrix} A^c_\nu(p) \\ \sqrt{2}\phi^{cd}_\nu(p) \end{pmatrix} , \tag{4.6}
$$

gives a matrix $\mathbf{X}^{abcd}$ that can be used to derive propagators for the mixing fields $\{A^a_\mu, \phi^{ab}_\mu, \bar{\phi}^{ab}_\mu\}$, [40],[41]. Referring to the quadratic part of the momentum space Lagrangian (4.5), it is easy to see that

$$
\mathbf{X}^{abcd} = \begin{pmatrix} -\delta^{ac} \left[p^2 \eta_{\mu\nu} - \left(1 - \frac{1}{\alpha}\right) p_\mu p_\nu \right] & +\gamma^2 f^{acd} \eta_{\mu\nu} \\ -\gamma^2 f^{cab} \eta_{\mu\nu} & -p^2 \delta^{ac} \delta^{bd} \eta_{\mu\nu} \end{pmatrix} . \tag{4.7}
$$

The tree-level propagators are derived by inverting the matrix $\mathbf{X}^{abcd}$ although the presence of Lorentz and colour tensor structures mean that it is not possible to invert $\mathbf{X}^{abcd}$ using the standard formula

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{[ad - bc]} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} . \tag{4.8}
$$

The inversion is achieved by making reference to a unit matrix with the appropriate tensor structure,

$$
\mathbf{X}\mathbf{X}^{-1} = \begin{pmatrix} \delta^{ab} \eta_{\mu\nu} & 0 \\ 0 & \delta^{ac} \delta^{bd} \eta_{\mu\nu} \end{pmatrix} . \tag{4.9}
$$

The procedure is not unlike that for obtaining the propagator for the gauge field in Quantum Electrodynamics or Quantum Chromodynamics, see, for example, [42]. Unlike the derivation for a single gauge field, the presence of four free colour indices in the

$\{\phi_\mu^{ab}, \bar{\phi}_\mu^{ab}\}$ channel admits the possibility of several different tensor structures. Deriving mixed propagators in this way does involve numerous tensor manipulations, after a little algebra we arrive at the inverted matrix,

$$(\mathbf{X}^{abcd})^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} , \tag{4.10}$$

where

$$\begin{aligned} A &= -\delta^{ab}p^2 \left[\frac{P_{\mu\nu}(p)}{[(p^2)^2 + C_A\gamma^4]} + \frac{\alpha L_{\mu\nu}(p)}{[(p^2)^2 + \alpha C_A\gamma^4]} \right] \\ B &= -\frac{f^{abc}\gamma^2}{\sqrt{2}} \left[\frac{P_{\mu\nu}(p)}{[(p^2)^2 + C_A\gamma^4]} + \frac{\alpha L_{\mu\nu}(p)}{[(p^2)^2 + \alpha C_A\gamma^4]} \right] \\ C &= -\frac{f^{abc}\gamma^2}{\sqrt{2}} \left[\frac{P_{\mu\nu}(p)}{[(p^2)^2 + C_A\gamma^4]} + \frac{\alpha L_{\mu\nu}(p)}{[(p^2)^2 + \alpha C_A\gamma^4]} \right] \\ D &= -\frac{\delta^{ac}\delta^{bd}}{p^2}\eta_{\mu\nu} + \frac{f^{abe}f^{cde}\gamma^4}{p^2} \left[\frac{P_{\mu\nu}(p)}{[(p^2)^2 + C_A\gamma^4]} + \frac{\alpha L_{\mu\nu}(p)}{[(p^2)^2 + \alpha C_A\gamma^4]} \right] . \end{aligned} \tag{4.11}$$

The Lorentz structures are given using expressions

$$P_{\mu\nu}(p) = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \qquad L_{\mu\nu}(p) = \frac{p_\mu p_\nu}{p^2} , \tag{4.12}$$

to describe transverse and longitudinal parts respectively, also, the explicit factors include the quadratic Casimir operator of the adjoint representation C_A, where

$$f^{acd}f^{bcd} = C_A\delta^{ab} . \tag{4.13}$$

Propagators for the mixing fields, including an explicit α dependence, may now be read off directly from $(\mathbf{X}^{abcd})^{-1}$. Taking the Landau gauge limit $\alpha = 0$, we obtain

$$\begin{aligned} \langle A_\mu^a(p) A_\nu^b(-p) \rangle &= -\,\delta^{ab}p^2 \frac{P_{\mu\nu}(p)}{[(p^2)^2 + C_A\gamma^4]} \\ \langle A_\mu^a(p) \bar{\phi}_\nu^{bc}(-p) \rangle &= -\,\frac{f^{abc}\gamma^2}{\sqrt{2}} \frac{P_{\mu\nu}(p)}{[(p^2)^2 + C_A\gamma^4]} \\ \langle A_\mu^a(p) \phi_\nu^{bc}(-p) \rangle &= -\,\frac{f^{abc}\gamma^2}{\sqrt{2}} \frac{P_{\mu\nu}(p)}{[(p^2)^2 + C_A\gamma^4]} \\ \langle \phi_\mu^{ab}(p) \bar{\phi}_\nu^{cd}(-p) \rangle &= -\,\frac{\delta^{ac}\delta^{bd}}{p^2}\eta_{\mu\nu} + \frac{f^{abe}f^{cde}\gamma^4}{p^2} \frac{P_{\mu\nu}(p)}{[(p^2)^2 + C_A\gamma^4]} . \end{aligned} \tag{4.14}$$

The Feynman rules for the remaining propagators may be read off directly from the Lagrangian,

$$\begin{aligned} \langle c^a(p) \bar{c}^b(-p) \rangle &= -\frac{\delta^{ab}}{p^2} \\ \langle \psi^{iI}(p) \bar{\psi}^{jJ}(-p) \rangle &= i\delta^{ij}\delta^{IJ}\frac{\not{p}}{p^2} \\ \langle \omega_\mu^{ab}(p) \bar{\omega}_\nu^{cd}(-p) \rangle &= -\frac{\delta^{ac}\delta^{bd}\eta_{\mu\nu}}{p^2} . \end{aligned} \tag{4.15}$$

We conclude this section by noting that for large momenta associated with the ultraviolet sector of Euclidean Yang-Mills theory it is reasonable to consider that $p^2 \gg \gamma^2$. As such the gauge field propagator approaches that of the ordinary Euclidean Yang-Mills propagator

$$\langle A^a_\mu(p) A^b_\nu(-p)\rangle = -\,\delta^{ab}\frac{1}{p^2}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) , \tag{4.16}$$

and the theory has effectively de-coupled from the additional fields used to implement the Gribov horizon condition. All of the familiar properties of ultraviolet QCD, for example asymptotic freedom, are unaltered.

Having derived all of the Feynman rules for the Gribov-Zwanziger Lagrangian it is possible to begin considering what implications the additional fields and mixing terms have for the infrared dynamics of Euclidean Yang-Mills theory. Applying the Euler-Lagrange equation for the field $\bar{\phi}^{ab}_\mu$ to (4.1)

$$\frac{\partial \mathcal{L}}{\partial \bar{\phi}} - \frac{\partial}{\partial x_\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\phi})}\right] = 0 , \tag{4.17}$$

results in an expression for the field ϕ^{ab}_μ given by,

$$\phi^{ab}_\mu = \frac{\gamma^2}{\sqrt{2}} f^{abc} \frac{1}{\partial_\nu D_\nu} A^c_\mu . \tag{4.18}$$

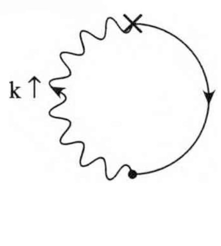
Using equation (4.18) it is possible to reproduce Zwanziger's formulation of the Gribov gap equation, [30],

$$\left\langle A^a_\mu \frac{1}{\partial_\nu D_\nu} A^a_\mu \right\rangle = \frac{d N_A}{C_A g^2} \tag{4.19}$$

in terms of the local expression,

$$f^{abc}\langle A^a_\mu \phi^{bc}_\mu\rangle = \frac{d N_A \gamma^2}{\sqrt{2} g^2} . \tag{4.20}$$

Having established the Feynman rule for the mixed propagator $\langle A^a_\mu(p)\phi^{bc}_\nu(-p)\rangle$, (4.14), it is now straightforward to compute the local gap equation (4.20) by closing the mixed $A^a_\mu \phi^{bc}_\nu$ propagator using the metric tensor $\eta_{\mu\nu}$ to produce a vacuum bubble describing the vacuum expectation value (vev) of $A^a_\mu \phi^{bc}_\nu$. The vev is calculated using dimensional regularization



$$= -\eta_{\mu\nu} \int \frac{d^d k}{(2\pi)^d} \frac{f^{abc}\gamma^2}{\sqrt{2}} \frac{1}{[(k^2)^2 + C_A \gamma^4]} \left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) , \tag{4.21}$$

where, rewriting the propagator with Gribov structure using partial fractions

$$\frac{1}{[(p^2)^2 + C_A\gamma^4]} = \frac{1}{2iC_A\gamma^2}\left(\frac{1}{[p^2 - i\sqrt{C_A}\gamma^2]} - \frac{1}{[p^2 + i\sqrt{C_A}\gamma^2]}\right) , \tag{4.22}$$

it is elementary to evaluate (4.21) using master integrals. The values for the master integrals are simple to calculate, in $d = 4 - 2\epsilon$, they are

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{[k^2 + i\sqrt{C_A}\gamma^2]} = \frac{i\sqrt{C_A}\gamma^2}{(4\pi)^2}\left(-\frac{1}{\epsilon} - 1 + \frac{1}{2}\overline{\ln}(C_A\gamma^4)\right) + O(\epsilon) , \tag{4.23}$$

and its complex conjugate, leading to the explicit, divergent, result

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{f^{abc}\gamma^2}{\sqrt{2}} \frac{(1-d)}{[(k^2)^2 + C_A\gamma^4]} = \frac{f^{abc}\gamma^2}{(4\pi)^2}\left(\frac{3}{\sqrt{2}\epsilon} + \frac{1}{\sqrt{2}} - \frac{3}{2\sqrt{2}}\overline{\ln}(C_A\gamma^4)\right) + O(\epsilon) . \tag{4.24}$$

The usual factor of $4\pi e^{-\bar{\gamma}}$, where $\bar{\gamma}$ is the Euler-Mascheroni constant, is absorbed into the regularization parameter μ^2, such that

$$\ln\left(\frac{C_A\gamma^4}{\mu^4}\right) \equiv \overline{\ln}(C_A\gamma^4) . \tag{4.25}$$

Divergences are removed using minimal subtraction (MS). Renormalization constants within the MS-scheme are independent of dimensionful parameters such as mass or momenta. They have the common structure,

$$Z(a) = 1 + \sum_{n=1}^{\infty} \frac{z^{(n)}(a)}{\epsilon^n} , \tag{4.26}$$

where $a = g^2/16\pi^2$ and g is the strong coupling constant of Yang-Mills theory. The explicit form of the constants (4.26) are obtained by multiplicative renormalization of specific Green's functions according to the prescription

$$\Gamma_r(a_r) = Z(a_r)\Gamma_o(a_o) , \tag{4.27}$$

where, a subscript r denotes a renormalized Green's function or quantity and, similarly, a subscript o a bare (unrenormalized) object. Renormalization of the Gribov-Zwanziger Lagrangian to a specific loop order within the minimal subtraction scheme is achieved by identifying a number of renormalization constants, with a structure (4.26), which combine to remove infinities that occur in solutions to the Green's functions derived using dimensional regularization. This amounts to the identification of a suitable renormalization constant for each field and coupling appearing in (4.1). These are,

$$\begin{aligned} A^a_{\mu\,\mathrm{o}} &= \sqrt{Z_A}\,A^a_\mu \ , \quad c^a_\mathrm{o} = \sqrt{Z_c}\,c^a \ , \quad \phi^{ab}_{\mu\,\mathrm{o}} = \sqrt{Z_\phi}\,\phi^{ab}_\mu \ , \quad \omega^{ab}_{\mu\,\mathrm{o}} = \sqrt{Z_\omega}\,\omega^{ab}_\mu \\ \psi_\mathrm{o} &= \sqrt{Z_\psi}\psi \ , \quad g_\mathrm{o} = Z_g\,g \ , \quad \gamma_\mathrm{o} = Z_\gamma\,\gamma \ . \end{aligned} \tag{4.28}$$

As discussed in chapter 3, using the recursive counterterm construction technique [33] it was shown in [21] that, remarkably, the Zwanziger ghost fields are renormalized by the same multiplicative factor as the Faddeev-Popov ghosts,

$$Z_c \;=\; Z_\phi \;=\; Z_\omega \;=\; \frac{1}{Z_g \sqrt{Z_A}} \;. \tag{4.29}$$

Also, in our conventions, the Gribov parameter γ is renormalized using the constant

$$Z_\gamma \;=\; (Z_A Z_c)^{-1/4} \;, \tag{4.30}$$

hence, no new independent renormalization factors have been introduced using the Gribov-Zwanziger formalism. For completeness, we note that the gluon, quark, Faddeev-Popov ghost and Yang-Mills coupling renormalization constants are identical to those of ordinary QCD.

4.2 One loop gap equation and ghost enhancement

The Gribov gap equation defines the coupling γ, calculating this in terms of the vev of the mixed $A^a_\mu \phi^{bc}_\nu$ propagator and referring back to this term in the Lagrangian

$$\frac{\gamma^2}{\sqrt{2}} f^{abc} A^a_\mu \phi^{bc}_\mu \,,$$

it is instructive to consider the gap equation as a Green's function that defines γ. Multiplying the expression (4.24) by Z_γ and using the relation (4.20) gives the one loop $\overline{\text{MS}}$ renormalized gap equation,

$$1 = C_A \left[\frac{5}{8} - \frac{3}{8} \ln \left(\frac{C_A \gamma^4}{\mu^4} \right) \right] a + O(a^2) \,. \tag{4.31}$$

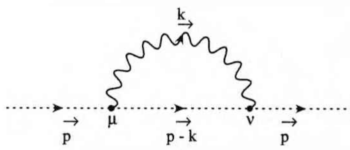
Having established the correct form of the one loop gap equation in the $\overline{\text{MS}}$ renormalization scheme, we turn our attention to the infrared behaviour of the Faddeev-Popov ghost two-point function. By expressing the Faddeev-Popov (FP) ghost propagator in the form

$$\langle c^a(p) \bar{c}^b(-p) \rangle \;=\; -\, \frac{\delta^{ab}}{p^2[1 + u(p^2)]} \tag{4.32}$$

it is possible to compare the behaviour of the Faddeev-Popov ghost propagator in the limit $p^2 \to 0$ with the Kugo-Ojima confinement criterion, where a full BRST analysis of QCD predicts that

$$1 \,+\, u(0) \;=\; 0 \;, \tag{4.33}$$

is a necessary condition for colour confinement in the Landau gauge, [19]. Here, $u(p^2)$, represents radiative corrections to the tree level propagator. In terms of the Landau gauge Feynman rules (4.14), the one loop correction to the FP ghost propagator is given by the expression



$$= -\int \frac{d^dk}{(2\pi)^d} \frac{\delta^{ab} p_\mu (p-k)_\nu}{(k-p)^2} \frac{k^2 P_{\mu\nu}(k)}{[(k^2)^2 + C_A\gamma^4]} \ . \tag{4.34}$$

From (4.34), it is easy to see that, at one loop order

$$p^2 u(p^2) = \int \frac{\mathrm{d}^dk}{(2\pi)^d} \frac{(k\cdot p)^2 - k^2p^2}{(k-p)^2[(k^2)^2 + C_A\gamma^4]} \ . \tag{4.35}$$

Using the identity

$$kp = \frac{1}{2}(k^2 + p^2 - (k-p)^2) \ , \tag{4.36}$$

and some elementary manipulations of the type (4.22), the one loop ghost correction (4.35) reduces to a sum of integrals of the form (4.23), and

$$\begin{aligned} &\int_k \frac{1}{(k-p)^2[k^2 + i\sqrt{C_A}\gamma^2]} \\ &= \frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon} + 2 - \overline{\ln}\left(p^2 + i\sqrt{C_A}\gamma^2\right) + \frac{i\sqrt{C_A}\gamma^2}{p^2}\ln\left(\frac{i\sqrt{C_A}\gamma^2}{p^2 + i\sqrt{C_A}\gamma^2}\right)\right) + O(\epsilon) \ , \end{aligned} \tag{4.37}$$

together with its complex conjugate, where

$$\int_k = \int \frac{\mathrm{d}^dk}{(2\pi)^d} \ . \tag{4.38}$$

The exact expression for the one loop ghost correction is given by

$$\begin{aligned} p^2u(p^2) &= -\frac{p^2g^2}{(4\pi)^2}\left(\frac{3}{4\epsilon} + \frac{5}{4} - \frac{3}{8}\overline{\ln}(C_A\gamma^4) - \frac{3}{8}\ln\left(1 + \frac{(p^2)^2}{C_A\gamma^4}\right)\right) \\ &\quad -\frac{g^2}{(4\pi)^2}\left(\frac{3}{4}\sqrt{C_A}\gamma^2\tan^{-1}\left(\frac{\sqrt{C_A}\gamma^2}{p^2}\right) - \frac{3\pi}{8}\sqrt{C_A}\gamma^2\right) \\ &\quad +\frac{(p^2)^2g^2}{(4\pi)^2}\left(\frac{1}{4\sqrt{C_A}\gamma^2}\tan^{-1}\left(\frac{\sqrt{C_A}\gamma^2}{p^2}\right)\right) \\ &\quad -\frac{g^2}{p^2(4\pi)^2}\left(\frac{C_A\gamma^4}{8}\ln\left(1 + \frac{(p^2)^2}{C_A\gamma^4}\right)\right) + O(\epsilon) \ . \end{aligned} \tag{4.39}$$

Expanding in p^2 using the relations

$$\begin{aligned} \ln\left(1 + \frac{(p^2)^2}{C_A\gamma^4}\right) &= \frac{(p^2)^2}{C_A\gamma^4} + O((p^2)^3) \\ \tan^{-1}\left(\frac{\sqrt{C_A}\gamma^2}{p^2}\right) &= \frac{\pi}{2} - \frac{p^2}{\sqrt{C_A}\gamma^2} + O((p^2)^2) \ , \end{aligned} \tag{4.40}$$

and removing divergences using the $\overline{\text{MS}}$ scheme leads to the finite expression

$$u(p^2) = C_A \left(\frac{3}{8} \overline{\ln}(C_A \gamma^4) - \frac{5}{8} \right) a + p^2 \frac{\sqrt{C_A} \pi}{8\gamma^2} a + (p^2)^2 \frac{1}{8\gamma^4} a + O((p^2)^3) \quad , (4.41)$$

in the limit $\epsilon \to 0$. Neglecting terms in p^2 and observing the gap equation (4.31) we note that the Kugo-Ojima criterion is satisfied at one-loop.

4.3 One loop corrections for the mixing fields

This section includes a review of one loop corrections to all the propagators comprised of the fields, $\theta \in \{A^a_\mu, \phi^{ab}_\mu, \bar{\phi}^{ab}_\mu\}$, calculated in [41]. Importantly, this work investigates one loop corrections to the gluon propagator to see if the zero momentum vanishing property of the tree level Gribov type propagator persists to higher orders in perturbation theory. Before turning attention to the one loop corrections to the mixed field propagators, it is instructive to describe the usual procedure for determining loop corrections to a field not involved in 2-point mixing. That is, calculating the corrections to the 2-point function and inverting the resulting expression which is then truncated at the appropriate order in the coupling constant. Exactly what was demonstrated above for the Faddeev-Popov ghost propagator. For the fields involved in 2-point mixing it is first necessary to calculate all of the 2-point function one loop corrections. Those four objects comprise a 2×2 matrix of one loop corrections analogous to the tree level Feynman rules of the matrix (4.7) read off directly using the quadratic part of the Lagrangian, (4.5). As with the tree level propagators, the one loop corrections to the 2-point mixing propagators are calculated by inverting the matrix of one loop corrections and making reference to an identity matrix with the appropriate tensor structure. An additional complication when inverting the one loop corrections stems from the need to account for the group theory structure of the $\phi^{ab}_\mu \bar{\phi}^{cd}_\nu$ 2-point function correctly. The gluon and mixed 2-point functions have the same structure as the Lagrangian term. As each ϕ^{ab}_μ field carries two colour indices, the one loop correction to the 2-point function requires four index objects denoted by the set $\{a, b, c, d\}$. A suitable basis of independent rank four objects describing all possible one loop corrections that may be generated by this set is given by

$$\{\delta^{ac}\delta^{bd}, \delta^{ad}\delta^{bc}, \delta^{ab}\delta^{cd}, f^{ace}f^{bde}, f^{abe}f^{cde}, d_A^{abcd}\} \ , \tag{4.42}$$

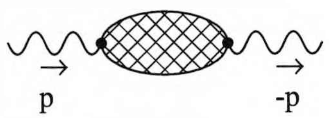
where

$$d_A^{abcd} = \frac{1}{6} \mathrm{tr} \left(T^a_A T^{(b}_A T^c_A T^{d)}_A \right) \quad , \tag{4.43}$$

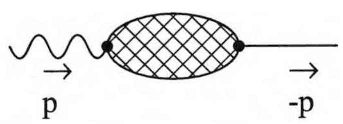
is the totally symmetric trace tensor and and $(T^a_A)_{bc} = -i f^{abc}$ is the adjoint representation representation of the colour group generators, [74]. Additional technical difficulty in computing the one loop corrections to the mixing fields make it preferable to work in the Landau gauge throughout, $\alpha = 0$, and factor out the common transverse projection operator $P_{\mu\nu}$. At one loop, the Landau gauge matrix of 2-point functions, with respect to the basis, $\{\frac{1}{\sqrt{2}}A^a_\mu, \phi^{ab}_\mu\}$, is given by

$$\begin{pmatrix} p^2\delta^{ac} & +\gamma^2 f^{acd} \\ -\gamma^2 f^{cab} & -p^2\delta^{ac}\delta^{bd} \end{pmatrix} + \begin{pmatrix} X\delta^{ac} & U f^{acd} \\ M f^{cab} & Q\delta^{ac}\delta^{bd} + W f^{ace} f^{bde} + R f^{abe} f^{cde} + S d_A^{abcd} \end{pmatrix} a \; + \; O(a^2) \, . \tag{4.44}$$

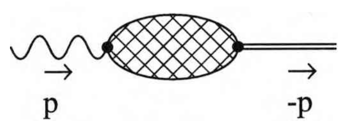
The quantities X, U, M, Q, W, R and S represent the one loop corrections such that

$X\delta^{ab} =$ 

$= \langle A^a_\mu(p) A^b_\nu(-p) \rangle$ (4.45)

$U f^{abc} =$ 

$= \langle A^a_\mu(p) \phi^{bc}_\nu(-p) \rangle$ (4.46)

$M f^{abc} =$ 

$= \langle A^a_\mu(p) \bar{\phi}^{bc}_\nu(-p) \rangle$ (4.47)

$\delta^{ac}\delta^{bd} + W f^{ace} f^{bde} + R f^{abe} f^{cde} + S d_A^{abcd}$

$=$ 

$= \langle \phi^{ab}_\mu(p) \bar{\phi}^{cd}_\nu(-p) \rangle \, .$ (4.48)

As such, up to one loop, the Landau gauge propagators take the form

$$\begin{pmatrix} \frac{p^2}{[(p^2)^2 + C_A\gamma^4]}\delta^{cp} & -\frac{\gamma^2}{[(p^2)^2 + C_A\gamma^4]} f^{cpq} \\ -\frac{\gamma^2}{[(p^2)^2 + C_A\gamma^4]} f^{pcd} & -\frac{1}{p^2}\delta^{cp}\delta^{dq} + \frac{\gamma^4}{p^2[(p^2)^2 + C_A\gamma^4]} f^{cdr} f^{pqr} \end{pmatrix} + \begin{pmatrix} A\delta^{cp} & C f^{cpq} \\ E f^{pcd} & G\delta^{cp}\delta^{dq} + J f^{cpe} f^{dqe} + K f^{cde} f^{pqe} + L d_A^{cdpq} \end{pmatrix} a \; + \; O(a^2) \tag{4.49}$$

where we have included the propagators from the previous section and the quantities A, C, E, G, J, K and L will depend on the one loop corrections defined in (4.44). Given these forms it is straightforward to check that

$$\begin{aligned} A &= -\frac{1}{[(p^2)^2 + C_A\gamma^4]^2} \\ &\quad \times \left[(p^2)^2 X - C_A\gamma^2 p^2 U - C_A\gamma^2 p^2 M + C_A\gamma^4 \left(Q + C_A R + \tfrac{1}{2} C_A W \right) \right] \end{aligned}$$

$$
\begin{aligned}
C &= \frac{1}{[(p^2)^2 + C_A\gamma^4]^2}\left[\gamma^2 p^2 X - C_A\gamma^4 M + (p^2)^2 U - \gamma^2 p^2\left(Q + C_A R + \tfrac{1}{2}C_A W\right)\right] \\
E &= \frac{1}{[(p^2)^2 + C_A\gamma^4]^2}\left[\gamma^2 p^2 X - C_A\gamma^4 U + (p^2)^2 M - \gamma^2 p^2\left(Q + C_A R + \tfrac{1}{2}C_A W\right)\right] \\
G &= -\frac{Q}{(p^2)^2} \quad , \quad J = -\frac{W}{(p^2)^2} \quad , \quad L = -\frac{S}{(p^2)^2} \\
K &= -\frac{1}{[(p^2)^2 + C_A\gamma^4]^2}\left[\gamma^4 X + \gamma^2 p^2 U + \gamma^2 p^2 M + (p^2)^2 R - \gamma^4\left(Q + \tfrac{1}{2}C_A W\right)\right] \\
&\quad + \frac{\gamma^4}{(p^2)^2[(p^2)^2 + C_A\gamma^4]}\left[Q + \tfrac{1}{2}C_A W\right]
\end{aligned}
\tag{4.50}
$$

by ensuring that the $O(a)$ term of the product of (4.44) and (4.49) vanishes.

4.4 Evaluation

Feynman diagrams describing the one loop corrections to each 2-point function are generated using the QGRAF package [43]. The complete set of Feynman rules described by the interaction Lagrangian (4.1) is fed into the QGRAF package purely in terms of occurrences of all the specific propagators and interaction vertices. No additional information about the nature of the propagators, beyond the commutation relations obeyed by the fields, or interaction vertices is necessary for the package to generate all the correct Feynman diagrams for a given Green's function at the desired loop order. For the gluon and ϕ^{ab}_μ corrections there are eight and two diagrams respectively and there are two diagrams for each of the mixed 2-point functions. A total of fourteen diagrams contribute to the two point correction matrix (4.44). The main purpose of the calculation is that of examining the $p^2 \to 0$ limit of the propagators to see if the vanishing property of the tree level gluon propagator is stable against radiative corrections. The calculation proceeds by evaluating resulting integrals exactly as functions of p^2 using a master integral of the form

$$
I_1(p, m_1^2, m_2^2; \alpha, \beta) = \int_k \frac{1}{[k^2 + m_1^2]^\alpha [(k-p)^2 + m_2^2]^\beta} \,. \tag{4.51}
$$

where the mass arguments take any combination of values in the set

$$
m_i^2 \in \{0, i\sqrt{C_A}\gamma^2, -i\sqrt{C_A}\gamma^2\} \,. \tag{4.52}
$$

Individual Feynman diagrams are reduced to a sum of graphs that are fully described by a cofactor of a (5.6) type integral using a computer algorithm written in the symbolic manipulation language FORM, [44]. The computer algorithm takes QGRAF output and furnishes each diagram with explicit details of the propagator and vertex Feynman rules using the appropriate Lorentz and colour indices. The next step organizes all of the group theory factors associated with a one loop calculation. Propagators with a

Gribov structure are written as a product of *standard* propagators

$$\frac{1}{[(q^2)^2 + C_A\gamma^4]} = \frac{1}{[q^2 + i\sqrt{C_A}\gamma^2][q^2 - i\sqrt{C_A}\gamma^2]} , \tag{4.53}$$

and then individual Feynman diagrams are converted into a sum of graphs with the desired structure using a systematic operation of identities with the general form

$$\begin{aligned}
\frac{q^2}{[q^2+m^2]} &= 1 - \frac{m^2}{[q^2+m^2]} \\
\frac{1}{q^2[q^2+m^2]} &= \frac{1}{m^2}\left(\frac{1}{q^2} - \frac{1}{[q^2+m^2]}\right) \\
\frac{1}{[q^2+m^2][q^2-m^2]} &= \frac{1}{2m^2}\left(\frac{1}{[q^2-m^2]} - \frac{1}{[q^2+m^2]}\right) ,
\end{aligned} \tag{4.54}$$

for generic m_i^2 in the set described above. Finally the algorithm combines results for all the diagrams contributing to a given 2-point correction into a single expression and multiplies the result by the appropriate renormalization constant. It is worth noting that in any final expression we derive the answer must be real, given that the initial integrals are real. This provides a useful internal check on the computation. Whilst the type of master integral, $I_1(p, m_1^2, m_2^2; 1, 1)$, has been studied and exploited many times we note that the key difference here is the presence of the complex mass. However, in the exact evaluation of our integrals we note that we use the formal results for $I_1(p, m_1^2, m_2^2; 1, 1)$ with real m_i^2 before analytically continuing to the values $\pm i\sqrt{C_A}\gamma^2$ we are interested in when the masses are non-zero. For completeness we note that the results for the two central integrals we use are, expanded to the finite parts,

$$\begin{aligned}
& I_1(p, i\sqrt{C_A}\gamma^2, i\sqrt{C_A}\gamma^2; 1, 1) \\
= \; & \frac{1}{\epsilon} + 2 - \ln\left(\frac{C_A\gamma^4}{\mu^4}\right) \\
& - \frac{1}{\sqrt{2}}\left[\left[\sqrt{\left(1 + \frac{16C_A\gamma^4}{(p^2)^2}\right)} + 1\right]^{1/2} \left[\frac{1}{2}\ln\left(\frac{16C_A\gamma^4}{(p^2)^2}\right)\right.\right. \\
& \qquad - \frac{1}{2}\ln\left[1 + \sqrt{(1 + \frac{16C_A\gamma^4}{(p^2)^2}}\right] \\
& \qquad \left. - \ln\left[\left(1 + \sqrt{\left(1 + \frac{16C_A\gamma^4}{(p^2)^2}\right)}\right)^{1/2} - \sqrt{2}\right]\right] \\
& + \left[\sqrt{\left(1 + \frac{16C_A\gamma^4}{(p^2)^2}\right)} - 1\right]^{1/2} \\
& \qquad \times \tan^{-1}\left[\sqrt{2}\left[\sqrt{\left(1 + \frac{16C_A\gamma^4}{(p^2)^2}\right)} - 1\right]^{-1/2}\right] \\
& - i\left[\sqrt{\left(1 + \frac{16C_A\gamma^4}{(p^2)^2}\right)} + 1\right]^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\times \tan^{-1}\left[\sqrt{2}\left[\sqrt{\left(1+\frac{16C_A\gamma^4}{(p^2)^2}\right)}-1\right]^{-1/2}\right] \\
&+ i\left[\sqrt{\left(1+\frac{16C_A\gamma^4}{(p^2)^2}\right)}-1\right]^{1/2}\left[\frac{1}{2}\ln\left(\frac{16C_A\gamma^4}{(p^2)^2}\right)\right. \\
&- \frac{1}{2}\ln\left[1+\sqrt{(1+\frac{16C_A\gamma^4}{(p^2)^2}}\right] \\
&- \left.\left.\ln\left[\left(1+\sqrt{\left(1+\frac{16C_A\gamma^4}{(p^2)^2}\right)}\right)^{1/2}-\sqrt{2}\right]\right]\right] + O(\epsilon)
\end{aligned}
\tag{4.55}
$$

and its conjugate, and

$$
\begin{aligned}
I_1(p, i\sqrt{C_A}\gamma^2, -i\sqrt{C_A}\gamma^2; 1, 1) &= \frac{1}{\epsilon} + 2 - \frac{[p^2 - 2i\sqrt{C_A}\gamma^2]}{2p^2}\ln\left(\frac{i\sqrt{C_A}\gamma^2}{\mu^2}\right) \\
&- \frac{[p^2 + 2i\sqrt{C_A}\gamma^2]}{2p^2}\ln\left(\frac{-i\sqrt{C_A}\gamma^2}{\mu^2}\right) \\
&- \frac{\sqrt{[4C_A\gamma^4 - (p^2)^2]}}{p^2}\tan^{-1}\left[-\frac{\sqrt{[4C_A\gamma^4 - (p^2)^2]}}{p^2}\right] \\
&+ O(\epsilon) \,.
\end{aligned}
\tag{4.56}
$$

When one of the masses m_i^2 is zero, we use the result

$$
\begin{aligned}
\int_k \frac{1}{k^2[(k-p)^2 + i\sqrt{C_A}\gamma^2]} &= \frac{1}{\epsilon} + 2 - \ln\left(\frac{i\sqrt{C_A}\gamma^2}{\mu^2}\right) \\
&- \frac{[p^2 + i\sqrt{C_A}\gamma^2]}{p^2}\ln\left[\frac{[p^2 + i\sqrt{C_A}\gamma^2]}{i\sqrt{C_A}\gamma^2}\right] + O(\epsilon)
\end{aligned}
\tag{4.57}
$$

and its conjugate. As one might expect, given the rather involved form of the master integrals (4.55) and (4.56), the exact form of each 2-point correction is rather involved and serves no purpose in describing the zero momentum behaviour of the one loop correction to the gluon propagator. The limit $p^2 \to 0$ is investigated by expanding in powers of p^2, that part of the 2-point corrections resulting from integrals including at least one massive propagator. Massless Feynman integrals have a predetermined momentum dependence $(p^2)^\Gamma$ where Γ is the dimension of the integral, for example,

$$
\int_k \frac{1}{k^2(k-p)^2} = \frac{1}{\epsilon}\left(1 + \epsilon(2 - \overline{\ln}(p^2))\right) + O(\epsilon) \,. \tag{4.58}
$$

Massless integrals are present in this calculation resulting either from propagators with a massless element or being generated by one of the operations (4.54) used in the algorithm. The presence of such integrals mean that, in general, one loop corrections to the 2-point functions are not given by a simple Taylor expansion in p^2 and will include transcendental functions with an external momentum dependence. The p^2 expansion for each 2-point function is given by

$$\begin{aligned}
\langle A^a_\mu(-p)A^b_\nu(p)\rangle &= \delta^{ab}\left[p^2 + \left(\left(\frac{15}{32}\ln\left(\frac{p^2}{\mu^2}\right) + \frac{163}{192}\ln\left(\frac{C_A\gamma^4}{\mu^4}\right) - \frac{263}{144}\right.\right.\right. \\
&\quad \left.\left.\left. + \frac{57\pi}{64}\frac{\sqrt{C_A}\gamma^2}{p^2}\right) C_A + \left(\frac{20}{9} - \frac{4}{3}\ln\left(\frac{p^2}{\mu^2}\right)\right) T_F N_f\right) p^2 a + O(a^2)\right] \\
&\quad + O\left((p^2)^2\right) \\
\langle A^a_\mu(-p)\bar{\phi}^{bc}_\nu(p)\rangle &= f^{abc}\left[1 + \left(\frac{31\pi}{192}\frac{\sqrt{C_A}p^2}{\gamma^2}\right) a + O(a^2)\right]\gamma^2 + O\left((p^2)^2\right) \\
\langle \phi^{ab}_\mu(-p)\bar{\phi}^{cd}_\nu(p)\rangle &= \left[\delta^{ac}\delta^{bd}\left[1 - \left(\frac{5}{8} - \frac{3}{8}\ln\left(\frac{C_A\gamma^4}{\mu^4}\right)\right) a\right] p^2\right. \\
&\quad \left. + \frac{3}{64}f^{ace}f^{bde}p^2 a + \frac{1}{24}f^{abe}f^{cde}p^2 a + \frac{9}{32}d_A^{abcd}\frac{p^2}{C_A}a + O(a^2)\right] \\
&\quad + O\left((p^2)^2\right) \qquad (4.59)
\end{aligned}$$

which determine X, M, U, Q, W, R and S, [41]. From (4.59) it is possible to deduce several interesting properties of the $p^2 \to 0$ limit. First, it is clear that at one loop the gluon *propagator*, (4.50), vanishes as the momentum vanishes. This is because when one substitutes the explicit values for X, M, U, Q, W, R and S from (4.59) into the expression for A in (4.50) then one finds, [41],

$$A = \left[\frac{p^2}{C_A\gamma^4}\left[\frac{3}{8}\ln\left(\frac{C_A\gamma^4}{\mu^4}\right) - \frac{215}{384}\right] a + O\left((p^2)^2\right)\right] + O(a^2) \,. \qquad (4.60)$$

4.5 Two loop gap equation and ghost enhancement

Continuing with the review of $\overline{\text{MS}}$ loop results for the Gribov-Zwanziger scenario, one loop corrections to the gap equation and Faddeev-Popov ghost propagator have been extended to two loop order using massless quarks in [40]. Part of the original work that will be presented here extends the analysis of [40] to include a consideration of massive quarks. The extended work is closely related to the original study and so the massless case is described in some detail first. The appropriate two loop corrections are generated using the QGRAF package, 19 diagrams contributing to the two loop extension for the vacuum expectation value (4.21), and 25 diagrams for the two loop ghost correction analogous to (4.34). Starting with the two loop gap equation, diagrams generated by the QGRAF package are furnished with all of the appropriate Feynman rules for the Lagrangian (4.1) and treated with a computer algorithm written in the symbolic manipulation language FORM, using identities with general form (4.54) and tensor reduction formulae such as,

$$\int_k f(k^2)k_\mu k_\nu = \eta_{\mu\nu}\Xi$$

$$\begin{aligned}
\eta_{\mu\nu}\int_k f(k^2)k_\mu k_\nu &= \int_k f(k^2)k^2 = d\Xi \Rightarrow \\
\Xi &= \frac{1}{d}\int_k f(k^2)k^2 \Rightarrow \\
\int_k f(k^2)k_\mu k_\nu &= \frac{\eta_{\mu\nu}}{d}\int_k f(k^2)k^2 \ ,
\end{aligned} \tag{4.61}$$

and the extension to four or more indices. Contractions between two different loop momenta are treated using the identity

$$kl = \frac{1}{2}(k^2 + l^2 - (k-l)^2) \ , \tag{4.62}$$

reducing all the components of the generated Feynman diagrams to numerator products with dot-products of single loop momenta k or l only, and denominator products of the massless and Gribov type propagators,

$$\begin{aligned}
prz_i^n &= \left(\frac{1}{q_i^2}\right)^n \\
prp_i^n &= \left(\frac{1}{q_i^2 + i\sqrt{C_A}\gamma^2}\right)^n \\
prm_i^n &= \left(\frac{1}{q_i^2 - i\sqrt{C_A}\gamma^2}\right)^n \ ,
\end{aligned} \tag{4.63}$$

where

$$q_i \in \{k, l, k-l\} \ . \tag{4.64}$$

In the computer algorithm, single loop momenta dot products are treated as inverse massless propagators, $(1/k^2)^{-1}$ and $(1/l^2)^{-1}$. This is done so that corrections described by the list of Feynman diagrams may be calculated using the minimum number of well known integral solutions. Having isolated all incidences of the loop momenta generated by the Feynman rules for the two loop corrections and restricted them to a limited set of expressions given in terms of the propagator types (4.63), it is possible to combine products of them into expressions I_2, with general form

$$I_2 = \texttt{int2(a1,a2,a3;b1,b2,b3;c1,c2,c3)} \ . \tag{4.65}$$

In this notation, the arguments in `a` refer to massless propagators, `b` positive Gribov width, `c` negative Gribov width and the indices 1, 2 and 3 refer to the momenta k, l and $(k-l)$ respectively, for example,

$$\begin{aligned}
&\texttt{int2(1,0,0;0,2,0;0,0,1)} \\
&= \int_{kl} \frac{1}{k^2[l^2 + i\sqrt{C_A}\gamma^2]^2[(k-l)^2 - i\sqrt{C_A}\gamma^2]} \ .
\end{aligned} \tag{4.66}$$

By employing a marker system, it is possible to perform simultaneously, elementary algebraic manipulations, on restricted groups of the integral functions I_2. A systematic treatment reduces each diagram to an expression given in terms of scalar two loop master integrals with a simple form,

$$I_2(m_1^2, m_2^2, m_3^2, a, b, c) = \int_{kl} \frac{1}{[k^2 + m_1^2]^a [l^2 + m_2^2]^b [(k-l)^2 + m_3^2]^c} , \qquad (4.67)$$

where the mass arguments take any combination of values

$$m \in \{0, i\sqrt{C_A}\gamma^2, -i\sqrt{C_A}\gamma^2\} , \qquad (4.68)$$

and the complete calculation is described by a set where the powers a, b and c take a value 0, 1 or 2.

Having described the procedure used to bring the two loop corrections to the gap equation into a manageable form, it is instructive to examine exactly what has been achieved up to this point. This is best done by looking at the FORM output describing the full two loop corrected gap equation treated using the computer algebra routine described above, before the integral values are put in.

```
gap =
      + ggr^2*Ca * (
       - 5/8
       + 3/8*lnbar(gribr^4*Ca)
       )
      + ggr^4*ep^-2*Ca^2 * (
       + 235/256
       )
      + ggr^4*ep^-2*Tf*Ca*Nf * (
       - 1/2
       )
      + ggr^4*ep^-1*Ca^2 * (
       + 339/256
       - 235/256*lnbar(gribr^4*Ca)
       )
      + ggr^4*ep^-1*Tf*Ca*Nf * (
       - 1/2
       + 1/2*lnbar(gribr^4*Ca)
       )
      + ggr^4*ep^-1*Ca^2* (
       + 35/768
```

```
 + 235/256*z2
 + 23/768*lnbar(gribr^4*Ca)
 + 95/512*lnbar(gribr^4*Ca)^2
 - 1251/4096*int2(0,0,0,0,0,0,2,1,1)
 + 15/512*int2(0,0,0,0,0,1,2,1,0)
 + 15/512*int2(0,0,0,2,1,0,0,0,1)
 - 1251/4096*int2(0,0,0,2,1,1,0,0,0)
 + 9/128*int2(0,0,1,2,0,0,0,1,0)
 + 9/128*int2(1,0,0,0,1,0,0,0,2)
 - 103/1024*int2(1,1,0,0,0,0,0,0,2)
 - 103/1024*int2(1,1,0,0,0,2,0,0,0)
 + 5/64*int2(2,0,0,0,1,0,0,0,1)
 - 5/256*int2(2,1,0,0,0,0,0,0,1)
 - 5/256*int2(2,1,0,0,0,1,0,0,0)
 )
+ ggr^2*Ca^2*pi^2 * (
 - 543/2048
 )
+ ggr^4*Tf*Ca*Nf * (
 + 1/24
 - 1/2*z2
 - 1/12*lnbar(gribr^4*Ca)
 - 1/8*lnbar(gribr^4*Ca)^2
 + 1/4*int2(1,1,0,0,0,0,0,0,2)
 + 1/4*int2(1,1,0,0,0,2,0,0,0)
 )
+ ggr^4*Tf*Ca*Nf*pi^2 * (
 + 1/8
 )
+ ggr^4*gribr^4*Ca^3 * (
 + 1/1024*int2(2,2,0,0,0,0,0,0,2)
 + 1/1024*int2(2,2,0,0,0,2,0,0,0)
 )
```

```
+ ggr^4*gribr^-2*Ca * (
 + 405/4096*(Ca)^(1/2)*int2(0,0,0,0,0,0,1,1,1)
 + 391/2048*(Ca)^(1/2)*int2(0,0,0,0,0,1,1,1,0)
 - 391/2048*(Ca)^(1/2)*int2(0,0,0,1,1,0,0,0,1)
 - 405/4096*(Ca)^(1/2)*int2(0,0,0,1,1,1,0,0,0)
 + 103/1024*(Ca)^(1/2)*int2(1,1,0,0,0,0,0,0,1)
 - 103/1024*(Ca)^(1/2)*int2(1,1,0,0,0,1,0,0,0)
 )
+ ggr^4*gribr^-2*Tf*Nf * (
 - 1/4*(Ca)^(1/2)*int2(1,1,0,0,0,0,0,0,1)
 + 1/4*(Ca)^(1/2)*int2(1,1,0,0,0,1,0,0,0)
 )
+ ggr^4*gribr^2*Ca^2 * (
 + 5/128*(Ca)^(1/2)*int2(0,0,2,1,0,0,0,2,0)
 - 5/128*(Ca)^(1/2)*int2(0,0,2,2,0,0,0,1,0)
 - 5/512*(Ca)^(1/2)*int2(2,1,0,0,0,0,0,0,2)
 + 5/512*(Ca)^(1/2)*int2(2,1,0,0,0,2,0,0,0)
 + 3/1024*(Ca)^(1/2)*int2(2,2,0,0,0,0,0,0,1)
 - 3/1024*(Ca)^(1/2)*int2(2,2,0,0,0,1,0,0,0)
 )
+ 1
 ;
```
(4.69)

In the notation of the computer output, the renormalized Yang-Mills coupling constant is denoted by `ggr`, and the renormalized Gribov parameter by `gribr`. The symbols, `Ca`,`Tf` and `Nf`, denote respectively, the quadratic Casimir operator in the adjoint representation, the trace of generators in the fundamental representation,

$$(T^a_F T^b_F)_{ii} = \delta^{ab} T_F \quad , \tag{4.70}$$

and the number of quark flavours. The explicit form of the two loop gap equation is calculated by directing the output (4.69) to an extensive library of two loop massive integrals, see, for example [46],[47],[48], stored electronically in a FORM module. The two loop correction to the Gribov gap equation, using massless quarks, was calculated to be [40],

$$1 \;=\; C_A \left[\frac{5}{8} \;-\; \frac{3}{8} \ln\left(\frac{C_A \gamma^4}{\mu^4} \right) \right] a$$

$$
\begin{aligned}
&+ \left[C_A^2\left(\frac{3893}{1536} - \frac{22275}{4096}s_2 + \frac{29}{128}\zeta(2) - \frac{65}{48}\ln\left(\frac{C_A\gamma^4}{\mu^4}\right)\right.\right. \\
&+ \left.\frac{35}{128}\left(\ln\left(\frac{C_A\gamma^4}{\mu^4}\right)\right)^2 + \frac{411}{1024}\sqrt{5}\zeta(2) - \frac{1317\pi^2}{4096}\right) \\
&+ C_AT_FN_f\left(-\frac{25}{24} - \zeta(2) + \frac{7}{12}\ln\left(\frac{C_A\gamma^4}{\mu^4}\right)\right. \\
&- \left.\left.\frac{1}{8}\left(\ln\left(\frac{C_A\gamma^4}{\mu^4}\right)\right)^2 + \frac{\pi^2}{8}\right)\right]a^2 , \qquad (4.71)
\end{aligned}
$$

where $\zeta(2)$ is the Riemann-Zeta function with argument 2, and $s_2 = \frac{9\sqrt{3}}{4}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)$, where Cl_2 is Clausen's function which is discussed shortly. Two loop corrections to the ghost propagator are calculated using a similar process. The propagator type master integrals

$$
J_2(p, m_1^2, m_2^2, m_3^2, a, b, c) \ , \qquad (4.72)
$$

where p refers to the external momentum, are derived using a vacuum bubble expansion in which the p dependent propagators are expanded recursively using the relation

$$
\frac{1}{[(k-p)^2+m^2]} = \frac{1}{[k^2+m^2]} + \frac{2kp-p^2}{[k^2+m^2][(k-p)^2+m^2]} \ . \qquad (4.73)
$$

The bubble expansion is truncated at the desired order in the external momentum such that

$$
J_2(p, m_1^2, m_2^2, m_3^2, a, b, c) \propto f(p^2)I_2(m_1^2, m_2^2, m_3^2, a, b, c) \ . \qquad (4.74)
$$

After evaluating the 25 diagrams that describe the two loop correction to the Faddeev-Popov ghost propagator in the Gribov-Zwanziger Lagrangian, the two-loop vacuum bubble expansion is compared with the gap equation and was found to coincide exactly, [40]. As such, ghost enhancement is demonstrated explicitly at two loop order in the $\overline{\text{MS}}$ scheme.

4.6 Two loop study with massive quarks

Extending the two loop study of the Gribov gap equation and ghost enhancement to include the possibility of arbitrarily massive quarks, it is necessary to study the available solutions to two loop massive integrals. Solution methods described in the literature are all derived using real mass arguments. Standard propagators generated by the Gribov-Zwanziger Lagrangian are defined by a complex width, not a real mass. An important and useful check on all of the calculations performed here is that all of the objects considered, namely corrections to 2-point functions or vacuum expectation values, must be given in terms of real quantities, in spite of the fact that we appeal to an imaginary width in the calculation process. When calculating Feynman diagrams including a complex width only, with no real mass present, it is possible to describe a complex width by an arbitrary real mass parameter and, at the end of the calculation, perform an analytical continuation such as,

$$\begin{aligned} m_1^2 &\to +i\sqrt{C_A}\gamma^2 \\ m_2^2 &\to -i\sqrt{C_A}\gamma^2 \ . \end{aligned} \tag{4.75}$$

Including the possibility of an arbitrary quark mass means that one of the diagrams used to evaluate the two loop corrections to the mixed propagator vacuum expectation value and ghost propagator includes both complex widths and a real mass. In this case we must recognize that the Gribov type propagators are complex during the integration process. In order to evaluate these we make use of an elegant solution, [46], to the integral

$$I_2(m_1^2, m_2^2, m_3^2, 1, 1, 1) = \int_{kl} \frac{1}{[k^2 + m_1^2][l^2 + m_2^2][(k-l)^2 + m_3^2]} , \tag{4.76}$$

where

$$\begin{aligned} (4\pi)^4 I_2(x, y, z, 1, 1, 1) &= -\frac{c}{2\epsilon^2} - \frac{1}{\epsilon}\left[\frac{3c}{2} - \mathrm{L}_1\right] - \frac{1}{2}\Big[\mathrm{L}_2 - 6\mathrm{L}_1 \\ &\quad + \xi(x,y,z) + c\,(7 + \zeta(2)) + (y + z - x)\,\overline{\ln}(y)\overline{\ln}(z) \\ &\quad + (z + x - y)\,\overline{\ln}(z)\overline{\ln}(y) + (y + x - z)\,\overline{\ln}(y)\overline{\ln}(x)\Big] \ . \end{aligned} \tag{4.77}$$

The elements of (4.77) are defined by the formulae

$$\begin{aligned} \mathrm{L}_i &= x\overline{\ln}^i(x) + y\overline{\ln}^i(y) + z\overline{\ln}^i(z) \\ c &= x + y + z \\ a &= \frac{1}{2}\left[x^2 + y^2 + z^2 - 2xy - 2xz - 2yz\right]^{1/2} \ . \end{aligned} \tag{4.78}$$

The value $a^2 = 0$, describes a cone in x, y, z space with apex at the origin. The integral is defined using two different regions, inside the cone, $a^2 \leq 0$, such that

$$\xi(x,y,z) = 8b\left[\mathrm{L}(\theta_x) + \mathrm{L}(\theta_y) + \mathrm{L}(\theta_z) - \frac{\pi}{2}\ln 2\right] , \tag{4.79}$$

where, $b^2 = -a^2$, and $\mathrm{L}(t)$ is Lobachevskij's function

$$\mathrm{L}(t) = -\int_0^t dx \ln(\cos x) \ . \tag{4.80}$$

Also,

$$\theta_x = \tan^{-1}\left[\frac{c - 2x}{2b}\right] \ . \tag{4.81}$$

The second solution is valid for the region outside the cone, $a^2 > 0$, such that

$$\xi(x,y,z) = 8a\left[\mathrm{M}(\phi_z) + \mathrm{M}(\phi_y) - \mathrm{M}(-\phi_x)\right] , \tag{4.82}$$

where

$$\begin{aligned} \mathrm{M}(t) &= -\int_0^t \mathrm{d}\phi \ln\left(\sinh(\phi)\right) \\ \phi_x &= \coth^{-1}\left[\frac{c-2x}{2a}\right] . \end{aligned} \tag{4.83}$$

Applying this formula to the Gribov type propagators, it is instructive to consider the possibility of using both solutions to evaluate the integral

$$\begin{aligned} & I_2(i\sqrt{C_A}\gamma^2, i\sqrt{C_A}\gamma^2, i\sqrt{C_A}\gamma^2, 1, 1, 1) \\ = & \int_{kl} \frac{1}{[k^2 + i\sqrt{C_A}\gamma^2][l^2 + i\sqrt{C_A}\gamma^2][(k-l)^2 + i\sqrt{C_A}\gamma^2]} . \end{aligned} \tag{4.84}$$

It is possible to evaluate this integral by appealing to an arbitrary real mass parameter m, and then perform an analytical continuation, $m \rightarrow i\sqrt{C_A}\gamma^2$, at the end. That is, we begin by applying the first solution to the integral

$$I_2(m^2, m^2, m^2, 1, 1, 1) = \int_{kl} \frac{1}{[k^2 + m^2][l^2 + m^2][(k-l)^2 + m^2]} , \tag{4.85}$$

where

$$\begin{aligned} a^2 &= -\frac{3m^4}{4} \\ c &= 3m^2 , \end{aligned} \tag{4.86}$$

such that

$$\begin{aligned} I_2(m^2, m^2, m^2, 1, 1, 1) &= -\frac{1}{2\epsilon^2}\left(3m^2\right) - \frac{1}{\epsilon}\left(\frac{9}{2}m^2 - 3m^2\overline{\ln}\left(m^2\right)\right) \\ &\quad -\frac{1}{2}\left[3m^2\left(\overline{\ln}\left(m^2\right)\right)^2 - 18m^2\overline{\ln}\left(m^2\right)\xi\left(m^2, m^2, m^2\right)\right. \\ &\quad \left. + 3m^2\left(7 + \zeta(2)\right) + 3m^2\left(\overline{\ln}\left(m^2\right)\right)^2\right] . \end{aligned} \tag{4.87}$$

According to the prescription, for $a^2 \leq 0$,

$$\begin{aligned} \xi(m^2, m^2, m^2) &= 8b\left[3\mathrm{L}(\theta_m) - \frac{\pi}{2}\ln 2\right] , \\ b &= \frac{\sqrt{3}m^2}{2} \\ \theta_m &= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} . \end{aligned} \tag{4.88}$$

A convenient solution to Lobachevskij's function is given in terms of Clausen's integral, [49],

$$\begin{aligned} \mathrm{L}(\theta_m) &= -\int_0^{\theta_m} \mathrm{d}x \ln\left((\cos x)\right) \\ \mathrm{L}(\tau) &= -\frac{1}{2}\mathrm{Cl}_2(\pi - 2\tau) + \tau \ln 2 , \end{aligned} \tag{4.89}$$

where

$$\mathrm{Cl}_2(\theta) = -\int_0^\theta \ln\left[2\left|\sin\left(\frac{\phi}{2}\right)\right|\right] d\phi\,. \tag{4.90}$$

Applying this to our formula and using the notation of, [48],

$$\begin{aligned}
\xi(m^2, m^2, m^2) &= -6\sqrt{3}m^2\mathrm{Cl}_2\left(\frac{2\pi}{3}\right) \\
&= -4\sqrt{3}m^2\mathrm{Cl}_2\left(\frac{\pi}{3}\right) \;=\; -27m^2 s_2\,,
\end{aligned} \tag{4.91}$$

where $s_2 = \frac{9\sqrt{3}}{4}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)$, giving for our arbitrary mass parameter integral

$$\begin{aligned}
I_2(m^2, m^2, m^2, 1, 1, 1) &= -\frac{1}{2\epsilon^2}\left(3m^2\right) - \frac{1}{\epsilon}\left(\frac{9}{2}m^2 - 3m^2\overline{\ln}\left(m^2\right)\right) \\
&\quad -\frac{1}{2}\Big[3m^2\left(\overline{\ln}\left(m^2\right)\right)^2 - 18m^2\overline{\ln}\left(m^2\right) - 27m^2 s_2 \\
&\qquad + 3m^2\left(7 + \zeta(2)\right) + 3m^2\left(\overline{\ln}\left(m^2\right)\right)^2\Big]\,.
\end{aligned} \tag{4.92}$$

If we evaluate

$$\begin{aligned}
&I_2(i\sqrt{C_A}\gamma^2, i\sqrt{C_A}\gamma^2, i\sqrt{C_A}\gamma^2, 1, 1, 1) \\
&= \int_{kl} \frac{1}{[k^2 + i\sqrt{C_A}\gamma^2][l^2 + i\sqrt{C_A}\gamma^2][(k-l)^2 + i\sqrt{C_A}\gamma^2]}\,,
\end{aligned} \tag{4.93}$$

directly, not appealing to an arbitrary real mass parameter, from (4.78) we see that $a^2 > 0$, and according to the prescription we are directed towards the second equation (4.82) which permits a solution, [50], given in terms of the dilogarithm, [51],

$$\begin{aligned}
\mathrm{M}(u) &= -\int_0^u d\phi \ln\left(\sinh(\phi)\right) \\
&= u\ln(2) - \frac{u^2}{2} + \mathrm{Li}_2(1) + \mathrm{Li}_2(-1) - \mathrm{Li}_2(e^{-u}) - \mathrm{Li}_2(-e^{-u})\,, \\
&= u\ln(2) - \frac{u^2}{2} + \frac{\zeta(2)}{2} - \mathrm{Li}_2(e^{-u}) - \mathrm{Li}_2(-e^{-u})\,,
\end{aligned} \tag{4.94}$$

where the dilogarithm is defined by

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-x)}{x} dx\,. \tag{4.95}$$

The properties of the dilogarithm with complex argument are well known, this solution is particularly useful when considering the case where we must include the complex structure of the Gribov type propagators explicitly. When we come to do this, we will find that the complex arguments derived in terms of a real arbitrary quark mass and an imaginary Gribov width are given by complicated expressions. As such, manipulations carried out on dilogarithms given by complicated function arguments are quite difficult to follow. The underlying simplicity of the identity operations for the complex dilogarithm are obscured when we come to evaluate them. Given this, we take the opportunity to introduce the dilogarithm of complex argument by evaluating (4.93) using (4.94) and showing that the final result is identical to an analytic continuation of the solution (4.92). Applying the formula (4.77) directly to integral (4.93), we obtain

$$
\begin{aligned}
I_2(i\sqrt{C_A}\gamma^2, i\sqrt{C_A}\gamma^2, i\sqrt{C_A}\gamma^2, 1,1,1) &= -\frac{1}{2\epsilon^2}\left(3i\sqrt{C_A}\gamma^2\right) \\
&\quad -\frac{1}{\epsilon}\left(\frac{9}{2}i\sqrt{C_A}\gamma^2 - 3i\sqrt{C_A}\gamma^2\overline{\ln}\left(i\sqrt{C_A}\gamma^2\right)\right) \\
&\quad -\frac{1}{2}\left(3i\sqrt{C_A}\gamma^2\left(\overline{\ln}\left(i\sqrt{C_A}\gamma^2\right)\right)^2\right. \\
&\quad - 18i\sqrt{C_A}\gamma^2\overline{\ln}\left(i\sqrt{C_A}\gamma^2\right) \\
&\quad + \xi\left(i\sqrt{C_A}\gamma^2, i\sqrt{C_A}\gamma^2, i\sqrt{C_A}\gamma^2\right) \\
&\quad + 3i\sqrt{C_A}\gamma^2\left(7+\zeta(2)\right) \\
&\quad \left. + 3i\sqrt{C_A}\gamma^2\left(\overline{\ln}\left(i\sqrt{C_A}\gamma^2\right)\right)^2\right) \quad , \qquad (4.96)
\end{aligned}
$$

where clearly the two solutions only differ by the ξ term. To show that this yields a result identical to an analytic continuation $m^2 \to i\sqrt{C_A}\gamma^2$ of (4.92), we use

$$
\xi(i\sqrt{C_A}\gamma^2, i\sqrt{C_A}\gamma^2, i\sqrt{C_A}\gamma^2) = 8a\left[2\mathrm{M}(t) - \mathrm{M}(s)\right] \ , \qquad (4.97)
$$

where,

$$
\begin{aligned}
a &= \frac{\sqrt{3}\sqrt{C_A}\gamma^2}{2} \\
c &= 3i\sqrt{C_A}\gamma^2 \\
t &= \coth^{-1}\left(\frac{i}{\sqrt{3}}\right) \\
s &= -\coth^{-1}\left(\frac{i}{\sqrt{3}}\right) \ . \qquad (4.98)
\end{aligned}
$$

Defining the inverse hyperbolic co-tangent using the logarithmic identity

$$
\coth^{-1}(z) = \frac{1}{2}\ln\left[\frac{z+1}{z-1}\right] \ , \qquad (4.99)
$$

we obtain a simple solution for s and t given by angles in the complex plane,

$$
\begin{aligned}
t &= \frac{1}{2}\ln\left[\frac{i+\sqrt{3}}{i-\sqrt{3}}\right] = \frac{1}{2}\ln(-1-i\sqrt{3}) - \frac{1}{2}\ln 2 = \frac{i}{2}\tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) = -\frac{\pi i}{3} \\
s &= \frac{1}{2}\ln\left[\frac{i-\sqrt{3}}{i+\sqrt{3}}\right] = \frac{1}{2}\ln(-1+i\sqrt{3}) - \frac{1}{2}\ln 2 = \frac{i}{2}\tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = -\frac{2\pi i}{3} \ ,
\end{aligned}
\qquad (4.100)
$$

where it is necessary to trace out the angles in a clockwise direction to arrive at the desired result, which produces the negative signs. Putting these values into the solution (4.94)

$$
\begin{aligned}
\mathrm{M}(t) &= t\ln(2) - \frac{t^2}{2} + \frac{\zeta(2)}{2} - \mathrm{Li}_2\left(\exp\left(\frac{i\pi}{3}\right)\right) - \mathrm{Li}_2\left(-\exp\left(\frac{i\pi}{3}\right)\right) \\
\mathrm{M}(s) &= s\ln(2) - \frac{s^2}{2} + \frac{\zeta(2)}{2} - \mathrm{Li}_2\left(\exp\left(\frac{2i\pi}{3}\right)\right) - \mathrm{Li}_2\left(-\exp\left(\frac{2i\pi}{3}\right)\right) \ ,
\end{aligned}
\qquad (4.101)
$$

provides an opportunity to begin applying the dilogarithm formulae to functions described in terms of a simple complex angle. Using the identity, [51],

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(-x) = \frac{1}{2}\mathrm{Li}_2(x^2) \ , \tag{4.102}$$

we obtain

$$\begin{aligned} \mathrm{M}(t) &= t\ln(2) - \frac{t^2}{2} + \frac{\zeta(2)}{2} - \frac{1}{2}\mathrm{Li}_2\left[\exp\left(\frac{2\pi i}{3}\right)\right] \\ \mathrm{M}(s) &= s\ln(2) - \frac{s^2}{2} + \frac{\zeta(2)}{2} - \frac{1}{2}\mathrm{Li}_2\left[\exp\left(\frac{4\pi i}{3}\right)\right] \ , \end{aligned} \tag{4.103}$$

expressions for $\mathrm{M}(t)$ and $\mathrm{M}(s)$ given in terms of elementary functions and a dilogarithm of complex argument which separates into real and imaginary parts according to the formula, [51],

$$\mathrm{Li}_2(re^{i\theta}) = -\frac{1}{2}\int_0^r \frac{\ln(1-2x\cos\theta + x^2)}{x}\mathrm{d}x + i\int_0^r \arctan\left[\frac{y\sin\theta}{1-y\cos\theta}\right]\frac{\mathrm{d}y}{y} . \tag{4.104}$$

The real part describes a dilogarithm defined using two variables,

$$\mathrm{Li}_2(r,\theta) = -\frac{1}{2}\int_0^r \frac{\ln(1-2x\cos\theta + x^2)}{x}\mathrm{d}x \ . \tag{4.105}$$

The imaginary part is defined using known functions

$$\begin{aligned} \int_0^r \arctan\left[\frac{y\sin(\theta)}{1-y\cos(\theta)}\right]\frac{\mathrm{d}y}{y} &= \omega\ln\frac{\sin(\omega)}{\sin(\omega+\theta)} \\ &\quad - \int_0^\omega [\ln(\sin(\phi)) - \ln(\sin(\phi+\theta))]\,\mathrm{d}\phi \\ &= \omega\ln(r) + \frac{1}{2}\mathrm{Cl}_2(2\omega) + \frac{1}{2}\mathrm{Cl}_2(2\theta) \\ &\quad - \frac{1}{2}\mathrm{Cl}_2(2\omega+2\theta) \ , \end{aligned} \tag{4.106}$$

where

$$\tan\omega = \frac{r\sin\theta}{1-r\cos\theta} \ . \tag{4.107}$$

Applying the formula (4.104),

$$\begin{aligned} \mathrm{M}(t) &= t\ln(2) - \frac{t^2}{2} + \frac{\zeta(2)}{2} - \frac{1}{2}\mathrm{Li}_2\left(1, \frac{2\pi}{3}\right) \\ &\quad -\frac{i}{2}\left[\omega_t\ln(1) + \frac{1}{2}\mathrm{Cl}_2(2\omega_t) + \frac{1}{2}\mathrm{Cl}_2(2\theta_t) - \frac{1}{2}\mathrm{Cl}_2(2\omega_t + 2\theta_t)\right] \\ \mathrm{M}(s) &= s\ln(2) - \frac{s^2}{2} + \frac{\zeta(2)}{2} - \frac{1}{2}\mathrm{Li}_2\left(1, \frac{4\pi}{3}\right) \ . \\ &\quad -\frac{i}{2}\left[\omega_s\ln(1) + \frac{1}{2}\mathrm{Cl}_2(2\omega_s) + \frac{1}{2}\mathrm{Cl}_2(2\theta_s) - \frac{1}{2}\mathrm{Cl}_2(2\omega_s + 2\theta_s)\right] \ , \end{aligned} \tag{4.108}$$

where

$$\begin{aligned} \theta_t &= \frac{2\pi}{3} & \theta_s &= \frac{4\pi}{3} \\ \omega_t &= \frac{\pi}{6} & \omega_s &= \frac{11\pi}{6} \ . \end{aligned} \tag{4.109}$$

Putting these values into the expressions gives,

$$\begin{aligned}
\mathrm{M}(t) &= t\ln(2) - \frac{t^2}{2} + \frac{\zeta(2)}{2} - \frac{1}{2}\mathrm{Li}_2\left(1, \frac{2\pi}{3}\right) \\
&\quad - \frac{i}{2}\left[\frac{\pi}{6}\ln(1) + \frac{1}{2}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{1}{2}\mathrm{Cl}_2\left(\frac{4\pi}{3}\right) - \frac{1}{2}\mathrm{Cl}_2\left(\frac{5\pi}{3}\right)\right] \\
\mathrm{M}(s) &= s\ln(2) - \frac{s^2}{2} + \frac{\zeta(2)}{2} - \frac{1}{2}\mathrm{Li}_2\left(1, \frac{4\pi}{3}\right) \;. \\
&\quad - \frac{i}{2}\left[\frac{11\pi}{6}\ln(1) + \frac{1}{2}\mathrm{Cl}_2\left(\frac{11\pi}{3}\right) + \frac{1}{2}\mathrm{Cl}_2\left(\frac{8\pi}{3}\right) - \frac{1}{2}\mathrm{Cl}_2\left(\frac{19\pi}{3}\right)\right] .
\end{aligned} \tag{4.110}$$

These equations are greatly simplified by the following formulae obeyed by the dilogarithm and Clausen's function

$$\begin{aligned}
\mathrm{Li}_2(1,\theta) &= \frac{1}{4}(\pi-\theta)^2 - \frac{\pi^2}{12} \\
\mathrm{Cl}_2(2n\pi \pm \theta) &= \mathrm{Cl}_2(\pm\theta) = \pm\,\mathrm{Cl}_2(\theta) \\
\mathrm{Cl}_2(\pi+\theta) &= -\,\mathrm{Cl}_2(\pi-\theta) \;,
\end{aligned} \tag{4.111}$$

leading to

$$\begin{aligned}
\mathrm{M}(t) &= t\ln(2) - \frac{t^2}{2} + \frac{\zeta(2)}{2} + \frac{\pi^2}{36} - \frac{i}{2}\left[\frac{\pi}{6}\ln(1) + \frac{2}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right] \\
\mathrm{M}(s) &= s\ln(2) - \frac{s^2}{2} + \frac{\zeta(2)}{2} + \frac{\pi^2}{36} - \frac{i}{2}\left[\frac{11\pi}{6}\ln(1) - \frac{2}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right] \;.
\end{aligned} \tag{4.112}$$

The explicit value of the Riemann-Zeta function with argument 2 is given by

$$\zeta(2) = \frac{\pi^2}{6} \;, \tag{4.113}$$

leading to further simplifications

$$\begin{aligned}
\mathrm{M}(t) &= -\frac{\pi i}{3}\ln(2) + \frac{\pi^2}{6} - \frac{i}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) \\
\mathrm{M}(s) &= -\frac{2\pi i}{3}\ln(2) + \frac{\pi^2}{3} + \frac{i}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) \;.
\end{aligned} \tag{4.114}$$

Putting these values for $\mathrm{M}(t)$ and $\mathrm{M}(s)$ back into our equation for ξ

$$\begin{aligned}
\xi(i\sqrt{C_A}\gamma^2, i\sqrt{C_A}\gamma^2, i\sqrt{C_A}\gamma^2) &= 8a\,[2\mathrm{M}(t) - \mathrm{M}(s)] \\
a &= \frac{\sqrt{3}\sqrt{C_A}\gamma^2}{2}
\end{aligned} \tag{4.115}$$

gives, finally

$$\xi(i\sqrt{C_A}\gamma^2, i\sqrt{C_A}\gamma^2, i\sqrt{C_A}\gamma^2) = -4\sqrt{3}i\sqrt{C_A}\gamma^2\mathrm{Cl}_2\left(\frac{\pi}{3}\right) = -\,27i\sqrt{C_A}\gamma^2 s_2 \;. \tag{4.116}$$

Applying the analytical continuation $m^2 \to i\sqrt{C_A}\gamma^2$ to (4.91), we see that

$$\xi(m^2, m^2, m^2) \;\to\; \xi(i\sqrt{C_A}\gamma^2, i\sqrt{C_A}\gamma^2, i\sqrt{C_A}\gamma^2), \tag{4.117}$$

and the two different solution methods give an identical result.

In this chapter, we have reviewed results for the mixed field propagators, $\{A^a_\mu, \phi^{ab}_\mu, \bar{\phi}^{ab}_\mu\}$, and seen explicitly at one loop in the $\overline{\text{MS}}$ scheme that the massive Gribov-Zwanziger action is renormalizable and does not introduce any additional divergences into Yang-Mills theory or indeed QCD, fixed in the Landau gauge, [41]. We have also given a review of two loop results for the gap equation and Faddeev-Popov ghost propagator including a brief consideration of the two-loop integral solutions used to evaluate them, [40]. In the next chapter, that two-loop study is extended to a consideration which includes the possibility of an arbitrary non-zero quark mass using a dilogarithm solution.

Chapter 5

Two Loop Mass Gap Equation With Massive Quarks

5.1 Integrals with massive quarks

Clearly in the real world quarks are not massless but massive. Therefore, to have a more realistic understanding of the Gribov situation it seems appropriate to include an arbitrary quark mass in the two loop $\overline{\text{MS}}$ gap equation and then see if the solution is consistent with the established behaviour of the ghost propagator at zero momentum. Diagrammatically, quarks appear for the first time at two loops in both calculations and, in both cases, quarks are present in just one diagram. Despite appearing in only one diagram, consideration of a massive quark greatly complicates the form taken by the final solution for the two loop gap equation.

We begin our consideration by defining the massive quark propagator to be

$$\langle \psi^{iI}(p) \overline{\psi}^{jJ}(-p) \rangle = \delta^{ij} \delta^{IJ} \frac{(\not{p} + i m_q)}{[p^2 + m_q^2]} . \tag{5.1}$$

Due to the difficulties associated with evaluating the integrals with massive quarks, we choose to evaluate that diagram separately. This simply requires including the additional propagators

$$prq^n = \left(\frac{1}{q^2 + m_q^2} \right)^n , \tag{5.2}$$

into the analysis outlined in the last chapter using (4.63) and (4.65). Proceeding in this way means that it is necessary to reconsider the tensor reduction formula (4.54) and define the integrals in the computer algebra modules with respect to a basis of twelve propagators, such that

$$I_2 = \texttt{int2(a1,a2,a3;b1,b2,b3;c1,c2,c3;d1,d2,d3)} , \tag{5.3}$$

where the arguments in d refer to the standard propagators with an arbitrary real quark mass (5.2). Whilst this looks complicated, it is necessary to capture all of the possible propagator combinations using a single expression. When we come to evaluate I_2, we will see that, as before, it is given by the standard two loop vacuum integrals

$$I_2(m_1^2, m_2^2, m_3^2, a, b, c) = \int_{kl} \frac{1}{[k^2+m_1^2]^a[(k-l)^2+m_2^2]^b[l^2+m_3^2]^c} , \tag{5.4}$$

where now,

$$m_i^2 \in \{0, i\sqrt{C_A}\gamma^2, -i\sqrt{C_A}\gamma^2, m_q^2\} . \tag{5.5}$$

Implementing the modified tensor reduction module produces a result containing four new integrals described in terms of massive quark propagators,

$$\begin{aligned}
I_2(m_q^2, m_q^2, i\sqrt{C_A}\gamma^2, 1, 1, 1) &= \int_{kl} \frac{1}{\left[k^2+m_q^2\right]\left[(k-l)^2+m_q^2\right][l^2+i\sqrt{C_A}\gamma^2]} \\
I_2(m_q^2, m_q^2, -i\sqrt{C_A}\gamma^2, 1, 1, 1) &= \int_{kl} \frac{1}{\left[k^2+m_q^2\right]\left[(k-l)^2+m_q^2\right][l^2-i\sqrt{C_A}\gamma^2]} \\
I_2(m_q^2, m_q^2, i\sqrt{C_A}\gamma^2, 1, 1, 2) &= \int_{kl} \frac{1}{\left[k^2+m_q^2\right]\left[(k-l)^2+m_q^2\right][l^2+i\sqrt{C_A}\gamma^2]^2} \\
I_2(m_q^2, m_q^2, -i\sqrt{C_A}\gamma^2, 1, 1, 2) &= \int_{kl} \frac{1}{\left[k^2+m_q^2\right]\left[(k-l)^2+m_q^2\right][l^2-i\sqrt{C_A}\gamma^2]^2} .
\end{aligned} \tag{5.6}$$

Beginning with the first integral

$$I_2(m_q^2, m_q^2, i\sqrt{C_A}\gamma^2) = \int_{kl} \frac{1}{\left[k^2+m_q^2\right]\left[(k-l)^2+m_q^2\right][l^2+i\sqrt{C_A}\gamma^2]} , \tag{5.7}$$

we apply the second solution (4.82) for the integral formula where $a^2 > 0$, such that

$$\xi(i\sqrt{C_A}\gamma^2, m_q^2, m_q^2) = 8a\left[2\mathrm{M}(\phi_{m_q^2}) - \mathrm{M}(-\phi_{i\sqrt{C_A}\gamma^2})\right] , \tag{5.8}$$

where

$$\begin{aligned}
a &= \frac{i}{2}\sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2} \\
c &= i\sqrt{C_A}\gamma^2 + 2m_q^2 .
\end{aligned} \tag{5.9}$$

These values for a and c give the intermediate variables

$$\begin{aligned}
\phi_{i\sqrt{C_A}\gamma^2} &= \coth^{-1}\left[\frac{-\sqrt{C_A}\gamma^2 - 2im_q^2}{\sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}\right] \\
\phi_{m_q^2} &= \coth^{-}1\left[\frac{\sqrt{C_A}\gamma^2}{\sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}\right] ,
\end{aligned} \tag{5.10}$$

which are re-expressed by applying the logarithmic definition

$$
\begin{aligned}
e^{-\phi_{i\sqrt{C_A}\gamma^2}} &= \frac{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}} \\
e^{-\phi_{m_q^2}} &= \sqrt{\frac{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}} \,.
\end{aligned}
\tag{5.11}
$$

Substituting these values into the expressions for the function $\mathrm{M}(u)$, (4.94),

$$
\begin{aligned}
\mathrm{M}(\phi_{m_q^2}) &= \frac{\zeta(2)}{2} + \frac{1}{2}\ln\left[\frac{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}\right]\ln(2) \\
&\quad -\frac{1}{8}\left[\ln\left[\frac{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}\right]\right]^2 \\
&\quad -\frac{1}{2}\mathrm{Li}_2\left[\frac{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}\right] ,
\end{aligned}
\tag{5.12}
$$

and

$$
\begin{aligned}
\mathrm{M}(-\phi_{i\sqrt{C_A}\gamma^2}) &= \frac{\zeta(2)}{2} + \ln\left[\frac{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}\right]\ln(2) \\
&\quad -\frac{1}{2}\left[\ln\left[\frac{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}\right]\right]^2 \\
&\quad -\mathrm{Li}_2\left[\frac{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}\right] \\
&\quad -\mathrm{Li}_2\left[\frac{\sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2} - \sqrt{C_A}\gamma^2}{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}\right] ,
\end{aligned}
\tag{5.13}
$$

where we have used the relationship [51],

$$
\mathrm{Li}_2(x) + \mathrm{Li}_2(-x) = \frac{1}{2}\mathrm{Li}_2(x^2) \,. \tag{5.14}
$$

This leads to the remarkably compact expression,

$$
\begin{aligned}
\xi(m_q^2, m_q^2, i\sqrt{C_A}\gamma^2) &= 4i\sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2} \\
&\quad \times \left[\frac{\zeta(2)}{2} + \frac{1}{4}\left[\ln\left[\frac{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}\right]\right]^2\right. \\
&\quad \left. +\mathrm{Li}_2\left[\frac{\sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2} - \sqrt{C_A}\gamma^2}{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}\right]\right] ,
\end{aligned}
\tag{5.15}
$$

giving the integral to the finite part

$$
\begin{aligned}
I_2(m_q^2, m_q^2, i\sqrt{C_A}\gamma^2) &= -\frac{1}{2\epsilon^2}\left(i\sqrt{C_A}\gamma^2 + 2m_q^2\right) \\
&- \frac{1}{\epsilon}\left(\frac{1}{2}(3i\sqrt{C_A}\gamma^2 + 6m_q^2) - 2m_q^2\overline{\ln}(m_q^2)\right. \\
&\left. - i\sqrt{C_A}\gamma^2\overline{\ln}(i\sqrt{C_A}\gamma^2)\right) \\
&- m_q^2(\overline{\ln}(m_q^2))^2 - \frac{1}{2}i\sqrt{C_A}\gamma^2(\overline{\ln}(i\sqrt{C_A}\gamma^2))^2 \\
&+ 6m_q^2\overline{\ln}(m_q^2) + 3i\sqrt{C_A}\gamma^2\overline{\ln}(i\sqrt{C_A}\gamma^2) \\
&- 2i\sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2} \\
&\times \left[\frac{\zeta(2)}{2} + \frac{1}{4}\left[\ln\left[\frac{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}\right]\right]^2\right. \\
&\left. + \mathrm{Li}_2\left[\frac{\sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2} - \sqrt{C_A}\gamma^2}{\sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2} + \sqrt{C_A}\gamma^2}\right]\right] \\
&- \frac{1}{2}\left(i\sqrt{C_A}\gamma^2 + 2m_q^2\right)(7 + \zeta(2)) \\
&- \frac{1}{2}\left(2m_q^2 - i\sqrt{C_A}\gamma^2\right)(\overline{\ln}(m_q^2))^2 \\
&- i\sqrt{C_A}\gamma^2\overline{\ln}(mq^2)\overline{\ln}(i\sqrt{C_A}\gamma^2) + O(\epsilon) \ . \qquad (5.16)
\end{aligned}
$$

We verify the validity of this solution by expanding all function arguments about the limit $m_q \to 0$ and check using the integral

$$
I_2(0,0,m^2) = \int_{kl} \frac{1}{k^2(k-l)^2\,[l^2 + m^2]} \ , \qquad (5.17)
$$

which admits a solution derived using elementary methods to give

$$
\begin{aligned}
I_2(0,0,m^2) &= -\frac{m^2}{2\epsilon^2} - \frac{m^2}{2\epsilon}\left(3 - 2\overline{\ln}(m^2)\right) \\
&-\frac{m^2}{2}\left(7 + 3\zeta(2) + 2\left(\overline{\ln}(m^2)\right)^2 - 6\overline{\ln}(m^2)\right) + O(\epsilon) \ . \qquad (5.18)
\end{aligned}
$$

Disregarding all terms proportional to m_q^2, $I_2(m_q^2, m_q^2, i\sqrt{C_A}\gamma^2)$ reduces to,

$$
\begin{aligned}
I_2(m_q^2, m_q^2, i\sqrt{C_A}\gamma^2) &= -\frac{i\sqrt{C_A}\gamma^2}{2\epsilon^2} - \frac{i\sqrt{C_A}\gamma^2}{\epsilon}\left(\frac{3}{2} - \overline{\ln}(i\sqrt{C_A}\gamma^2)\right) \\
&-\frac{1}{2}i\sqrt{C_A}\gamma^2(\overline{\ln}(i\sqrt{C_A}\gamma^2))^2 + 3i\sqrt{C_A}\gamma^2\overline{\ln}(i\sqrt{C_A}\gamma^2) \\
&- i\sqrt{C_A}\gamma^2\left[\zeta(2) + \frac{1}{2}\left[\left(\overline{\ln}(i\sqrt{C_A}\gamma^2)\right)^2\right.\right. \\
&\left.\left. -2\overline{\ln}(i\sqrt{C_A}\gamma^2)\overline{\ln}(m_q^2) + \left(\overline{\ln}(m_q^2)\right)^2\right] + 2\mathrm{Li}_2\left[\frac{im_q^2}{\sqrt{C_A}\gamma^2}\right]\right] \\
&-\frac{i\sqrt{C_A}\gamma^2}{2}(7 + \zeta(2)) + \frac{i\sqrt{C_A}\gamma^2}{2}(\overline{\ln}(m_q^2))^2 \\
&-i\sqrt{C_A}\gamma^2\overline{\ln}(mq^2)\overline{\ln}(i\sqrt{C_A}\gamma^2) + O(\epsilon) \ , \qquad (5.19)
\end{aligned}
$$

where we note that the imaginary dilogarithm vanishes as $m_q^2 \to 0$ and the remaining logarithmic terms in m_q^2 cancel. By making the analytic continuation $m^2 \to i\sqrt{C_A}\gamma^2$, we see that our integral is entirely consistent with $I_2(0,0,i\sqrt{C_A}\gamma^2)$ in the limit of zero quark mass

$$I_2(m_q^2, m_q^2, i\sqrt{C_A}\gamma^2) \quad \to \quad I_2(0,0,i\sqrt{C_A}\gamma^2)\,, \tag{5.20}$$

and it is understood that the right hand side is given by (5.18). Next we turn to the complex conjugate integral $I_2(m_q^2, m_q^2, -i\sqrt{C_A}\gamma^2)$ using the formula

$$\xi(-i\sqrt{C_A}\gamma^2, m_q^2, m_q^2) \;=\; 8a\left[2\mathrm{M}(\phi_{m_q^2}) - \mathrm{M}(-\phi_{-i\sqrt{C_A}\gamma^2})\right]\;, \tag{5.21}$$

where the new variables are now

$$\begin{aligned} a &= \frac{i}{2}\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2} \\ c &= 2m_q^2 - i\sqrt{C_A}\gamma^2\,. \end{aligned} \tag{5.22}$$

This leads to the intermediate variables

$$\begin{aligned} e^{-\phi_{-i\sqrt{C_A}\gamma^2}} &= \frac{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2}} \\ e^{-\phi_{m_q^2}} &= \sqrt{\frac{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2}}}\;, \end{aligned} \tag{5.23}$$

and putting these into our solution,

$$\begin{aligned} \xi(m_q^2, m_q^2, -i\sqrt{C_A}\gamma^2) &= 4i\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2} \\ &\times \left[\frac{\zeta(2)}{2} + \frac{1}{4}\left[\ln\left[\frac{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2}}\right]\right]^2\right. \\ &\left. + \mathrm{Li}_2\left[\frac{\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2} + \sqrt{C_A}\gamma^2}{\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2} - \sqrt{C_A}\gamma^2}\right]\right]\,. \end{aligned} \tag{5.24}$$

The second integral value is given by the expression

$$\begin{aligned} I_2(m_q^2, m_q^2, -i\sqrt{C_A}\gamma^2) &= -\frac{1}{2\epsilon^2}\left(2m_q^2 \;-\; i\sqrt{C_A}\gamma^2\right) \\ &- \frac{1}{\epsilon}\left(\frac{1}{2}(6m_q^2 - 3i\sqrt{C_A}\gamma^2) \;-\; 2m_q^2\overline{\ln}(m_q^2)\right. \\ &\quad \left. +\; i\sqrt{C_A}\gamma^2\overline{\ln}(-i\sqrt{C_A}\gamma^2)\right) \\ &- m_q^2(\overline{\ln}(m_q^2))^2 \;+\; \frac{1}{2}i\sqrt{C_A}\gamma^2(\overline{\ln}(-i\sqrt{C_A}\gamma^2))^2 \\ &+ 6m_q^2\overline{\ln}(m_q^2) \;-\; 3i\sqrt{C_A}\gamma^2\overline{\ln}(-i\sqrt{C_A}\gamma^2) \end{aligned}$$

$$
\begin{aligned}
&- 2i\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2} \\
&\times \left[\frac{\zeta(2)}{2} + \frac{1}{4}\left[\ln\left[\frac{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2}}\right]\right]^2\right. \\
&\left. \qquad + \mathrm{Li}_2\left[\frac{\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2} + \sqrt{C_A}\gamma^2}{\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2} - \sqrt{C_A}\gamma^2}\right]\right] \\
&- \frac{1}{2}\left(2m_q^2 - i\sqrt{C_A}\gamma^2\right)(7 + \zeta(2)) \\
&- \frac{1}{2}\left(2m_q^2 + i\sqrt{C_A}\gamma^2\right)(\overline{\ln}(m_q^2))^2 \\
&+ i\sqrt{C_A}\gamma^2\overline{\ln}(mq^2)\overline{\ln}(-i\sqrt{C_A}\gamma^2) + O(\epsilon)\,. \qquad (5.25)
\end{aligned}
$$

When we compare the value for $I_2(m_q^2, m_q^2, -i\sqrt{C_A}\gamma^2)$ with the result (5.16), it does not appear to describe its complex conjugate. Also, there is a potential singularity contained within the complex dilogarithm when we attempt to explore the zero quark mass limit. To resolve this problem, we turn to the properties of the dilogarithm and use the identity, [51],

$$
\mathrm{Li}_2(-1/x) + \mathrm{Li}_2(-x) = 2\mathrm{Li}_2(-1) - \frac{1}{2}(\ln(x))^2\,. \qquad (5.26)
$$

Equating the dilogarithm argument with $-1/x$,

$$
\frac{-1}{x} = \left[\frac{\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2} + \sqrt{C_A}\gamma^2}{\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2} - \sqrt{C_A}\gamma^2}\right], \qquad (5.27)
$$

the identity (5.26) yields

$$
\begin{aligned}
&\mathrm{Li}_2\left[\frac{\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2} + \sqrt{C_A}\gamma^2}{\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2} - \sqrt{C_A}\gamma^2}\right] \\
&= - \mathrm{Li}_2\left[\frac{\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2} - \sqrt{C_A}\gamma^2}{\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2} + \sqrt{C_A}\gamma^2}\right] - \zeta(2) \\
&\quad - \frac{1}{2}\left[\ln\left[\frac{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2} + \sqrt{C_A}\gamma^2}\right]\right]^2\,. \qquad (5.28)
\end{aligned}
$$

Putting this formula for the complex dilogarithm back into the evaluated integral gives

$$
\begin{aligned}
I_2(m_q^2, m_q^2, -i\sqrt{C_A}\gamma^2) &= -\frac{1}{2\epsilon^2}\left(2m_q^2 - i\sqrt{C_A}\gamma^2\right) \\
&\quad - \frac{1}{\epsilon}\left(\frac{1}{2}(6m_q^2 - 3i\sqrt{C_A}\gamma^2) - 2m_q^2\overline{\ln}(m_q^2)\right. \\
&\quad \left. + i\sqrt{C_A}\gamma^2\overline{\ln}(-i\sqrt{C_A}\gamma^2)\right)
\end{aligned}
$$

$$
\begin{aligned}
&- m_q^2(\overline{\ln}(m_q^2))^2 \;+\; \frac{1}{2}i\sqrt{C_A}\gamma^2(\overline{\ln}(-i\sqrt{C_A}\gamma^2))^2 \\
&+ 6m_q^2\overline{\ln}(m_q^2) \;-\; 3i\sqrt{C_A}\gamma^2\overline{\ln}(-i\sqrt{C_A}\gamma^2) \\
&+ 2i\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2} \times \\
&\left[\frac{\zeta(2)}{2} \;+\; \frac{1}{4}\left[\ln\left[\frac{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2}}\right]\right]^2\right. \\
&\left.+\mathrm{Li}_2\left[\frac{\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2} - \sqrt{C_A}\gamma^2}{\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2} + \sqrt{C_A}\gamma^2}\right]\right] \\
&- \frac{1}{2}\left(2m_q^2 \;-\; i\sqrt{C_A}\gamma^2\right)(7 + \zeta(2)) \\
&- \frac{1}{2}\left(2m_q^2 + i\sqrt{C_A}\gamma^2\right)(\overline{\ln}(m_q^2))^2 \\
&+ i\sqrt{C_A}\gamma^2\overline{\ln}(m_q^2)\overline{\ln}(-i\sqrt{C_A}\gamma^2) + O(\epsilon)\,. \qquad (5.29)
\end{aligned}
$$

Comparing this with the integral value, (5.16), we see that $I_2(m_q^2, m_q^2, -i\sqrt{C_A}\gamma^2)$ is equal to the complex conjugate of $I_2(m_q^2, m_q^2, i\sqrt{C_A}\gamma^2)$ as would of course be expected given the original form of the two integrals. This represents an important check on a correct implementation of the integration procedure and appropriate use of the identities used to manipulate the dilogarithm. Finally we note that in the zero quark mass limit our expression is entirely consistent with (5.18), for analytic continuation $m^2 \;\rightarrow\; -i\sqrt{C_A}\gamma^2$.

A solution to the second pair of integrals in (5.6), given in terms of the first pair is straightforward by observing that,

$$
\begin{aligned}
\frac{\partial}{\partial\gamma^2}I_2(m_q^2, m_q^2, i\sqrt{C_A}\gamma^2) &= \frac{\partial}{\partial\gamma^2}\int_p\int_q \frac{1}{\left[p^2 + m_q^2\right]\left[(p+q)^2 + m_q^2\right]\left[q^2 + i\sqrt{C_A}\gamma^2\right]} \\
&= -\int_{kl}\frac{i\sqrt{C_A}}{\left[k^2 + m_q^2\right]\left[(k-l)^2 + m_q^2\right]\left[l^2 + i\sqrt{C_A}\gamma^2\right]^2}\,.
\end{aligned}
\qquad (5.30)
$$

Hence, the second two integrals are defined using the relations

$$
\begin{aligned}
\overline{I}_2(m_q^2, m_q^2, i\sqrt{C_A}\gamma^2) &= \left(\frac{i}{\sqrt{C_A}}\right)\frac{\partial}{\partial\gamma^2}I_2(m_q^2, m_q^2, i\sqrt{C_A}\gamma^2) \\
\overline{I}_2(m_q^2, m_q^2, -i\sqrt{C_A}\gamma^2) &= \left(\frac{-i}{\sqrt{C_A}}\right)\frac{\partial}{\partial\gamma^2}I_2(m_q^2, m_q^2, -i\sqrt{C_A}\gamma^2)\,. \qquad (5.31)
\end{aligned}
$$

We note that

$$
\frac{\partial}{\partial\gamma^2}\mathrm{Li}_2(x) \;=\; -\frac{\partial x}{\partial\gamma^2}\frac{\ln(1-x)}{x}\,,
$$

where

$$
x \;=\; \left[\frac{\sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2} - \sqrt{C_A}\gamma^2}{\sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2} + \sqrt{C_A}\gamma^2}\right]\,.
$$

Given this, we arrive at the value

$$\begin{aligned}
\overline{I}_2(m_q^2, m_q^2, i\sqrt{C_A}\gamma^2) \;=\;& \frac{1}{2\epsilon^2} - \frac{1}{\epsilon}\left(-\frac{1}{2} + \overline{\ln}(i\sqrt{C_A}\gamma^2)\right) \\
& + \frac{1}{2}\left(\overline{\ln}(i\sqrt{C_A}\gamma^2)\right)^2 - 5\overline{\ln}(i\sqrt{C_A}\gamma^2) + \frac{1}{2} + \frac{\zeta(2)}{2} + 2i\pi \\
& - \frac{1}{2}\left(\overline{\ln}(m_q^2)\right)^2 + \overline{\ln}(m_q^2)\overline{\ln}(i\sqrt{C_A}\gamma^2) - 4\ln(2) \\
& + \left[\frac{\left(2\sqrt{C_A}\gamma^2 + 4im_q^2\right)\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A^2\gamma^8 + 16\sqrt{C_A}^2\gamma^4 m_q^4}}\right] \\
& \times \left[\frac{\zeta(2)}{2} + \frac{1}{4}\left[\ln\left[\frac{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}\right]\right]^2\right. \\
& + \left.\mathrm{Li}_2\left[\frac{\sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2} - \sqrt{C_A}\gamma^2}{\sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2} + \sqrt{C_A}\gamma^2}\right]\right] \\
& + 4\overline{\ln}\left[\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}\right] . \qquad (5.32)
\end{aligned}$$

Once again, we check this solution in the limit $m_q \to 0$ by referring to the integral

$$\overline{I}_2(0, 0, m^2) \;=\; \int_p \int_q \frac{1}{p^2(p+q)^2\left[q^2 + m^2\right]^2} , \qquad (5.33)$$

and find that for analytical continuation $m^2 \to i\sqrt{C_A}\gamma^2$

$$\begin{aligned}
\overline{I}_2(0, 0, m^2) \;=\;& \frac{1}{2\epsilon^2} + \frac{1}{2\epsilon}\left(1 - 2\overline{\ln}(m^2)\right) \\
& + \left(\overline{\ln}(m^2)\right)^2 - \overline{\ln}(m^2) + \frac{1}{2} + \frac{3\zeta(2)}{2} + O(\epsilon) . \qquad (5.34)
\end{aligned}$$

the solution $\overline{I}_2(0, 0, m^2)$, obtained using elementary methods, is consistent with the expression (5.32) in the limit of vanishing quark mass. Using a similar analysis, the integral $\overline{I}_2(m_q^2, m_q^2, -i\sqrt{C_A}\gamma^2)$ was shown to be given by the complex conjugate of the solution (5.32). This concludes our consideration of the solutions to the integrals (5.6).

5.2 Separation into real and imaginary parts

As discussed in chapter 4, the solution to the gap equation, indeed any quantity we are considering using the Gribov-Zwanziger model, must be given by a real quantity, regardless of the complex nature of the integrals used during evaluation. In order to convert the evaluated integrals (5.6) to a form suitable for use in the gap equation, they must be separated into real and imaginary parts, this begins by applying the elementary lemma for a complex variable $z = a + ib$,

$$\sqrt{a \pm ib} \;=\; \frac{1}{\sqrt{2}}\sqrt{\sqrt{a^2 + b^2} + a} \;\pm\; \frac{i}{\sqrt{2}}\sqrt{\sqrt{a^2 + b^2} - a} , \qquad (5.35)$$

where a and b are real. Applying this formula to the complex square roots in our integral values

$$\sqrt{C_A\gamma^4 \pm 4i\sqrt{C_A}\gamma^2 m_q^2} = \frac{1}{\sqrt{2}}\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}+C_A\gamma^4} \pm\frac{i}{\sqrt{2}}\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}-C_A\gamma^4}\,. \quad (5.36)$$

Given this we note that the logarithmic terms are separated into real and imaginary parts according to the standard formula

$$\ln(a\pm ib) = \frac{1}{2}\ln(a^2+b^2)\pm i\tan^{-1}\left(\frac{b}{a}\right) , \quad (5.37)$$

and recall that the complex dilogarithm is separated according to the formula, [51],

$$\mathrm{Li}_2(re^{i\theta}) = \mathrm{Li}_2(r,\theta)+i\left[\omega\ln(r)+\frac{1}{2}\mathrm{Cl}_2(2\omega)+\frac{1}{2}\mathrm{Cl}_2(2\theta)-\frac{1}{2}\mathrm{Cl}_2(2\omega+2\theta)\right] , \quad (5.38)$$

where

$$\tan\omega = \frac{r\sin\theta}{1-r\cos\theta} . \quad (5.39)$$

For our complex dilogarithm we obtain

$$\begin{aligned}
&\mathrm{Li}_2(re^{i\theta}) \\
&= \mathrm{Li}_2\left[\frac{\sqrt{C_A\gamma^4+4i\sqrt{C_A}\gamma^2 m_q^2}-\sqrt{C_A}\gamma^2}{\sqrt{C_A\gamma^4+4i\sqrt{C_A}\gamma^2 m_q^2}+\sqrt{C_A}\gamma^2}\right] \\
&= \mathrm{Li}_2\left[\frac{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}-C_A\gamma^4+i\sqrt{2}\sqrt{C_A}\gamma^2\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}-C_A\gamma^4}}{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}+C_A\gamma^4+\sqrt{2}\sqrt{C_A}\gamma^2\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}+C_A\gamma^4}}\right]
\end{aligned} \quad (5.40)$$

where, converting this into the form of a complex radial angle, we note that

$$r = \frac{\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}-C_A\gamma^4}}{\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}+C_A\gamma^4}} , \quad (5.41)$$

and

$$\tan\theta = \frac{\sqrt{2}\sqrt{C_A}\gamma^2}{\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}-C_A\gamma^4}} . \quad (5.42)$$

The secondary angle ω is calculated using, for example

$$(\tan\theta)^2 = \frac{2C_A\gamma^4}{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}-C_A\gamma^4} = \frac{1}{(\cos\theta)^2}-1 , \quad (5.43)$$

such that

$$\begin{aligned}
\cos\theta &= \sqrt{\frac{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}-C_A\gamma^4}{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}} \\
\sin\theta &= \frac{\sqrt{2}\sqrt{C_A}\gamma^2}{\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}} \,,
\end{aligned} \tag{5.44}$$

where for each function we have used the positive square root. Implementing this into the formula for $\tan\omega$, we find it is actually identical to the radius, (5.41),

$$\tan\omega = \frac{\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}-C_A\gamma^4}}{\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}} = r\,. \tag{5.45}$$

Given the rather long winded nature of these expressions we note that in what follows, it will often make sense to refer to them simply by name, that is, r, θ and ω, where

$$\omega = \tan^{-1}\left(\frac{\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}-C_A\gamma^4}}{\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}}\right)\,, \tag{5.46}$$

and

$$\theta = \tan^{-1}\left(\frac{\sqrt{2}\sqrt{C_A}\gamma^2}{\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}-C_A\gamma^4}}\right)\,. \tag{5.47}$$

Given this, we record that for the solution to the integral $I_2(m_q^2, m_q^2, i\sqrt{C_A}\gamma^2)$, separated into real and imaginary parts, we obtain

$$\begin{aligned}
I_2(m_q^2, m_q^2, i\sqrt{C_A}\gamma^2) &= -\frac{1}{2\epsilon^2}\left(i\sqrt{C_A}\gamma^2 + 2m_q^2\right) \\
&\quad - \frac{1}{\epsilon}\left(\frac{1}{2}(3i\sqrt{C_A}\gamma^2+6m_q^2) - 2m_q^2\overline{\ln}(m_q^2)\right. \\
&\quad \left. - i\sqrt{C_A}\gamma^2\overline{\ln}(i\sqrt{C_A}\gamma^2)\right) \\
&\quad - m_q^2(\overline{\ln}(m_q^2))^2 - \frac{1}{2}i\sqrt{C_A}\gamma^2(\overline{\ln}(i\sqrt{C_A}\gamma^2))^2 \\
&\quad + 6m_q^2\overline{\ln}(m_q^2) + 3i\sqrt{C_A}\gamma^2\overline{\ln}(i\sqrt{C_A}\gamma^2) \\
&\quad - \sqrt{2}i\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^2}+C_A\gamma^4} \\
&\quad \times\left[\frac{\zeta(2)}{2}+\frac{1}{4}\left[\frac{1}{2}\overline{\ln}\left(\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4\right)\right.\right. \\
&\quad \times\overline{\ln}\left(\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right) \\
&\quad -2\ln(2)+i\frac{\pi}{2}-\overline{\ln}(\sqrt{C_A}\gamma^2)-\overline{\ln}(m_q^2)
\end{aligned}$$

$$
\begin{aligned}
&+2i\tan^{-1}\left[\frac{\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}-C_A\gamma^4}}{\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}}\right]\Bigg]^2 \\
&+\mathrm{Li}_2(r,\theta)+i\omega\left[2\ln(2)+\frac{1}{2}\overline{\ln}(C_A\gamma^4)+\overline{\ln}(m_q^2)\right. \\
&-\frac{1}{2}\overline{\ln}\left(\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4\right) \\
&\left.-\overline{\ln}\left(\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right)\right] \\
&\left.+\frac{i}{2}\mathrm{Cl}_2(2\omega)+\frac{i}{2}\mathrm{Cl}_2(2\theta)-\frac{i}{2}\mathrm{Cl}_2(2\omega+2\theta)\right] \\
&+\sqrt{2}\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^2}-C_A\gamma^4} \\
&\times\left[\frac{\zeta(2)}{2}+\frac{1}{4}\left[\frac{1}{2}\overline{\ln}\left(\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4\right)\right.\right. \\
&\times\overline{\ln}\left(\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right) \\
&-2\ln(2)+i\frac{\pi}{2}-\overline{\ln}(\sqrt{C_A}\gamma^2)-\overline{\ln}(m_q^2) \\
&+2i\tan^{-1}\left[\frac{\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}-C_A\gamma^4}}{\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}}\right]\Bigg]^2 \\
&+\mathrm{Li}_2(r,\theta)+i\omega\left[2\ln(2)+\frac{1}{2}\overline{\ln}(C_A\gamma^4)+\overline{\ln}(m_q^2)\right. \\
&-\frac{1}{2}\overline{\ln}\left(\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4\right) \\
&\left.-\overline{\ln}\left(\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right)\right] \\
&\left.+\frac{i}{2}\mathrm{Cl}_2(2\omega)+\frac{i}{2}\mathrm{Cl}_2(2\theta)-\frac{i}{2}\mathrm{Cl}_2(2\omega+2\theta)\right] \\
&-\frac{1}{2}\left(i\sqrt{C_A}\gamma^2+2m_q^2\right)(7+\zeta(2)) \\
&-\frac{1}{2}\left(2m_q^2-i\sqrt{C_A}\gamma^2\right)(\overline{\ln}(m_q^2))^2 \\
&-i\sqrt{C_A}\gamma^2\overline{\ln}(m_q^2)\overline{\ln}(i\sqrt{C_A}\gamma^2)+O(\epsilon)\,. \qquad (5.48)
\end{aligned}
$$

The conjugate is now clearly of a similar form. Separating the remaining integral solutions into real and imaginary parts according to an analogous process, we put the appropriate form of the solutions to the integrals (5.6) back into the gap equation derived using massive quarks, which gives the result

$$
\begin{aligned}
1 &= aC_A\left[\frac{5}{8}-\frac{3}{8}\ln\left(\frac{C_A\gamma^4}{\mu^4}\right)\right] \\
&+a^2\left(\frac{\sqrt{C_A}T_FN_fm_q^2}{\gamma^2}\right)\left[4\omega+\frac{\pi}{2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+a^2\Bigg[C_A^2\Bigg[\frac{3893}{1536}-\frac{22275}{4096}s_2+\frac{29}{128}\zeta(2)-\frac{65}{48}\overline{\ln}(C_A\gamma^4)\\
&+\frac{35}{128}\left(\overline{\ln}(C_A\gamma^4)\right)^2+\frac{411}{1024}\sqrt{5}\zeta(2)-\frac{1317\pi^2}{4096}\Bigg]\\
&+C_AT_FN_f\Bigg[-\frac{25}{24}+2\ln(2)+\frac{1}{2}\left(\overline{\ln}(m_q^2)\right)^2-\frac{1}{2}\overline{\ln}(m_q^2)\overline{\ln}(C_A\gamma^4)\\
&+\frac{19}{12}\overline{\ln}(C_A\gamma^4)-\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]\\
&-\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]+\frac{\pi^2}{8}\Bigg]\Bigg]\\
&+a^2\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^2}+C_A\gamma^4}\left(\frac{\sqrt{C_A}T_FN_f}{\sqrt{2}\gamma^2}\right)\\
&\times\Bigg[-\frac{\zeta(2)}{4}-\frac{1}{2}\left(\ln(2)\right)^2-\frac{1}{2}\ln(2)\overline{\ln}(m_q^2)-\frac{1}{4}\ln(2)\ln(C_A\gamma^4)\\
&+\frac{1}{2}\ln(2)\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^2m_q^4}+C_A\gamma^4}\right]\\
&+\frac{1}{2}\ln(2)\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^2m_q^4}+C_A\gamma^4}\right]\\
&-\frac{1}{8}\left(\overline{\ln}(m_q^2)\right)^2-\frac{1}{8}\overline{\ln}(m_q^2)\overline{\ln}(C_A\gamma^4)-\frac{1}{32}\left(\overline{\ln}(C_A\gamma^4)\right)^2\\
&+\frac{1}{4}\overline{\ln}(m_q^2)\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]\\
&+\frac{1}{8}\overline{\ln}(C_A\gamma^4)\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]\\
&-\frac{1}{8}\left[\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]\right]^2\\
&+\frac{1}{4}\overline{\ln}(m_q^2)\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]\\
&+\frac{1}{8}\overline{\ln}(\sqrt{C_A}\gamma^2)\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]\\
&-\frac{1}{4}\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]\\
&\times\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]\\
&-\frac{1}{8}\left[\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]\right]^2\\
&+\frac{1}{2}\omega^2-\frac{1}{2}\mathrm{Li}_2(r,\theta)+\frac{\pi}{4}\omega+\frac{\pi^2}{32}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+a^2\left[\frac{\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^2}+C_A\gamma^4}}{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}}\right]\left(\frac{\sqrt{C_A}T_FN_fm_q^4}{\sqrt{2}\gamma^2}\right)\\
&\times\Big[-\zeta(2)-2\,(\ln(2))^2-2\ln(2)\overline{\ln}(m_q^2)-\ln(2)\overline{\ln}(C_A\gamma^4)\\
&+2\ln(2)\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]\\
&+2\ln(2)\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]\\
&-\frac{1}{2}\left(\overline{\ln}(m_q^2)\right)^2-\frac{1}{2}\overline{\ln}(m_q^2)\overline{\ln}(C_A\gamma^4)-\frac{1}{8}\left(\overline{\ln}(C_A\gamma^4)\right)^2\\
&+\overline{\ln}(m_q^2)\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]\\
&+\frac{1}{2}\overline{\ln}(C_A\gamma^4)\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]\\
&-\frac{1}{2}\left[\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]\right]^2\\
&+\overline{\ln}(m_q^2)\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]\\
&+\frac{1}{2}\overline{\ln}(C_A\gamma^4)\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]\\
&-\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]\\
&\times\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]\\
&-\frac{1}{2}\left[\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]\right]^2+2\omega^2-2\mathrm{Li}_2(r,\theta)+\pi\omega+\frac{\pi^2}{8}\Big]\\
&+a^2\left[\frac{\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^2}+C_A\gamma^4}}{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}}\right]\left(\frac{(C_A)^{3/2}T_FN_f\gamma^2}{\sqrt{2}}\right)\\
&\times\Big[-\frac{\zeta(2)}{4}-\frac{1}{2}\,(\ln(2))^2-\frac{1}{2}\ln(2)\overline{\ln}(m_q^2)-\frac{1}{4}\ln(2)\overline{\ln}(C_A\gamma^4)\\
&+\frac{1}{2}\ln(2)\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]\\
&+\frac{1}{2}\ln(2)\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]\\
&-\frac{1}{8}\left(\overline{\ln}(m_q^2)\right)^2-\frac{1}{8}\overline{\ln}(m_q^2)\overline{\ln}(C_A\gamma^4)-\frac{1}{32}\left(\overline{\ln}(C_A\gamma^4)\right)^2\\
&+\frac{1}{4}\overline{\ln}(m_q^2)\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]
\end{aligned}
$$

$$+\frac{1}{8}\overline{\ln}(C_A\gamma^4)\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]$$

$$-\frac{1}{8}\left[\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]\right]^2$$

$$+\frac{1}{4}\overline{\ln}(m_q^2)\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]$$

$$+\frac{1}{8}\overline{\ln}(C_A\gamma^4)\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]$$

$$-\frac{1}{4}\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]$$

$$\times\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]$$

$$-\frac{1}{8}\left[\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]\right]^2+\frac{\omega^2}{2}-\frac{1}{2}\mathrm{Li}_2(r,\theta)+\frac{\pi\omega}{4}+\frac{\pi^2}{32}\Bigg]$$

$$+a^2\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^2}-C_A\gamma^4}\left(\frac{\sqrt{C_A}T_FN_f}{\sqrt{2}\gamma^2}\right)$$

$$\times\left[-\frac{\pi}{4}\ln(2)-\frac{\pi}{8}\overline{\ln}(m_q^2)-\frac{\pi}{16}\overline{\ln}(C_A\gamma^4)\right.$$

$$+\frac{\pi}{8}\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]$$

$$+\frac{\pi}{8}\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]$$

$$+\frac{1}{4}\mathrm{Cl}_2(2\theta)-\frac{1}{4}\mathrm{Cl}_2(2\theta+2\omega)+\frac{1}{4}\mathrm{Cl}_2(2\omega)$$

$$+\left[\frac{m_q^4}{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}}\right]$$

$$\times\left[\pi\ln(2)+\frac{\pi}{2}\overline{\ln}(m_q^2)+\frac{\pi}{4}\overline{\ln}(C_A\gamma^4)\right.$$

$$-\frac{1}{2}\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]$$

$$-\frac{1}{2}\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]$$

$$-\mathrm{Cl}_2(2\theta)+\mathrm{Cl}_2(2\theta+2\omega)-\mathrm{Cl}_2(2\omega)]]$$

$$+a^2\left[\frac{\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^2}-C_A\gamma^4}}{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}}\right]\left(\frac{(C_A)^{3/2}\gamma^2T_FN_f}{\sqrt{2}}\right)$$

$$\times\left[-\frac{1}{4}\mathrm{Cl}_2(2\theta)+\frac{1}{4}\mathrm{Cl}_2(2\theta+2\omega)-\frac{1}{4}\mathrm{Cl}_2(2\omega)+\frac{\pi}{4}\ln(2)+\frac{\pi}{8}\overline{\ln}(m_q^2)\right.$$

$$+\frac{\pi}{16}\overline{\ln}(C_A\gamma^4)-\frac{\pi}{8}\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]$$

$$-\frac{1}{8}\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right]\Bigg] + O(a^3) \, . \tag{5.49}$$

Importantly, by inspection, the solution (5.49) for the two loop gap equation incorporating an arbitrary quark mass calculated in the $\overline{\text{MS}}$ renormalization scheme, is given by a real expression so that the first consistency check has been passed. As a further check that the calculation has been performed correctly, (5.49) must reduce to the previously established two loop gap equation, calculated using massless quarks, in the limit $m_q \to 0$. The necessary expansion formula are given by

$$\begin{aligned}
\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4} &= C_A\gamma^4+8m_q^4+O(m_q^8) \\
\frac{1}{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}} &= \frac{1}{C_A\gamma^4}-\frac{8}{C_A^2\gamma^8}m_q^4+O(m_q^8) \\
\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4} &= \sqrt{2}\sqrt{C_A}\gamma^2+\frac{2\sqrt{2}}{\sqrt{C_A\gamma^2}}m_q^4+O(m_q^8) \\
\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}-C_A\gamma^4} &= 2\sqrt{2}m_q^4+O(m_q^8) \ .
\end{aligned} \tag{5.50}$$

For the logarithmic terms we obtain,

$$\begin{aligned}
\overline{\ln}\left(\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right) &= \frac{1}{2}\ln(2)+\frac{1}{2}\overline{\ln}(C_A\gamma^2) \\
\overline{\ln}\left(\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4m_q^4}+C_A\gamma^4}\right) &= \frac{3}{2}\ln(2)+\frac{1}{2}\overline{\ln}(C_A\gamma^2) \ .
\end{aligned} \tag{5.51}$$

For the dilogarithm function arguments

$$\begin{aligned}
r &\to \frac{m_q^4}{\sqrt{C_A\gamma^2}} = O(m_q^4) \\
\omega &\to \tan^{-1}\left(\frac{m_q^4}{\sqrt{C_A\gamma^2}}\right) = O(m_q^4) \\
\theta &\to \tan^{-1}\left(\frac{\sqrt{C_A\gamma^2}}{2m_q^4}\right) \to \frac{\pi}{2} \, ,
\end{aligned} \tag{5.52}$$

leading to

$$\begin{aligned}
\mathrm{Li}_2(r,\theta) &\to \mathrm{Li}_2(0,\pi/2)=0 \\
\mathrm{Cl}_2(2\theta) &\to \mathrm{Cl}_2(\pi)=0 \\
\mathrm{Cl}_2(2\omega) &\to \mathrm{Cl}_2(0)=0 \\
\mathrm{Cl}_2(2\theta+2\omega) &\to \mathrm{Cl}_2(\pi)=0 \ .
\end{aligned} \tag{5.53}$$

Implementing this into the FORM computer algebra routine we recover the result

$$1 = C_A\left[\frac{5}{8}-\frac{3}{8}\ln\left(\frac{C_A\gamma^4}{\mu^4}\right)\right]a$$

$$
\begin{aligned}
&+ \left[C_A^2 \left(\frac{3893}{1536} - \frac{22275}{4096} s_2 + \frac{29}{128} \zeta(2) - \frac{65}{48} \ln \left(\frac{C_A \gamma^4}{\mu^4} \right) \right.\right. \\
&+ \left. \frac{35}{128} \left(\ln \left(\frac{C_A \gamma^4}{\mu^4} \right) \right)^2 + \frac{411}{1024} \sqrt{5} \zeta(2) - \frac{1317\pi^2}{4096} \right) \\
&+ C_A T_F N_f \left(-\frac{25}{24} - \zeta(2) + \frac{7}{12} \ln \left(\frac{C_A \gamma^4}{\mu^4} \right) \right. \\
&- \left.\left. \frac{1}{8} \left(\ln \left(\frac{C_A \gamma^4}{\mu^4} \right) \right)^2 + \frac{\pi^2}{8} \right) \right] a^2 \, , \qquad (5.54)
\end{aligned}
$$

which is identical to the two loop gap equation derived using massless quarks, as required. Extending the analysis to the two loop corrections for the Faddeev-Popov ghost 2-point function again evaluated using massive quark propagators, the two loop corrections coincide exactly with the expression (5.49) and so ghost enhancement with massive quarks is established formally at two loops in the $\overline{\text{MS}}$ renormalization scheme.

At the end of this thesis we will describe, briefly, new developments in the infrared sector of QCD derived using nonperturbative methods which appear to be inconsistent with results obtained using the Gribov-Zwanziger model. In particular, findings for the zero momentum behaviour of the gluon and ghost propagators are different from those described above. As we saw in chapters 2 and 3, a rigorous identification of the principal Gribov region contained within the boundary $\partial\Omega$ is a difficult task. Similarly implementing a suitable restriction into the generating functional of QCD is also challenging. A treatment restricted to Landau gauge Yang-Mills/QCD is necessary to overcome these difficulties. Despite this limitation, a motivation that the Faddeev-Popov gauge fixing prescription is to be improved so that only physically distinct gauge fields are considered by the functional integral remains valid. If the recent nonperturbative results for the gluon and Faddeev-Popov ghost propagators in the infrared sector are found to be correct, the Gribov-Zwanziger model can be modified using a *natural* extension. Modifying the GZ model in this way, preliminary studies show that results for the gluon and FP ghost propagators are consistent with new nonperturbative data. Since the extension involves incorporating a real mass for the localizing fields, $\{\phi_\mu^{ab}, \bar{\phi}_\mu^{ab}\}$, if preliminary studies for the extended GZ model propagators are to be expanded to a formal loop study like those described here and in chapter 4, a treatment using a dilogarithm solution will be necessary. As such, we believe that the techniques successfully employed here show that this is a possibility.

Chapter 6

Gauge Invariant Mass Operators

6.1 Candidate operators

We now turn to the question of a gluon mass in Yang-Mills theory, where, theoretical evidence suggests the possibility of a condensate with mass dimension two, [52],[53],[54],[55],[56]. In analogy with the preceding section, it is believed that condensates may play an important role in the infrared dynamics of Euclidean Yang-Mills theories. The remainder of this thesis is concerned with incorporating a suitable mass operator into the Euclidean Yang-Mills Lagrangian, with the aim of deriving a physically meaningful, gauge invariant, effective potential to investigate the possibility that a dynamically generated gluon mass in the infrared sector of QCD will reduce the vacuum energy of the theory. Much preliminary work incorporating weaker candidate operators into the Yang-Mills and QCD Lagrangians has already been done, producing encouraging data. We postpone a comprehensive review of that work until the discussion section of this thesis. A consideration of how mass operators are incorporated into non-Abelian gauge theories in such a way that it is possible to perform physically meaningful calculations is quite complicated. It is instructive to have a separate look at a significant class of mass operators and consider their strengths and weaknesses along with any similarities or differences. In this short chapter we give a review of mass operators originally presented, also as a review, as part of a larger paper, [58].

The ideal candidate for a physically meaningful mass operator is the gauge invariant condensate

$$A^2_{\text{min}} \equiv \min_{\{u\}} \frac{1}{2} \int d^4x A^{a\,u}_\mu A^{a\,u}_\mu = \min_{\{u\}} \operatorname{tr} \int d^4x A^u_\mu A^u_\mu \,, \tag{6.1}$$

where

$$A^u_\mu = u^\dagger A_\mu u + \frac{i}{g} u^\dagger \partial_\mu u \,, \tag{6.2}$$

is the gauge transform for the gluon field A_μ of non-Abelian gauge theory for an arbitrary semi-simple Lie group, and u is an element of the group. As we shall see, an explicit determination of an absolute minimum is a highly non trivial task.

We begin a discussion of mass with an account of non-local gauge invariant Abelian mass operators that can be added to the Maxwell action. In the Abelian case all of these operators return an identical expression when the classical equations of motion are employed; they are classically equivalent. All of the non-local Abelian mass operators can be generalized to the non-Abelian case although the feature of classical equivalence is no longer true in the non-Abelian case.

The most straightforward way to introduce a gauge invariant mass operator to the the Maxwell action is given by the operator,

$$\begin{aligned} \mathcal{O}_1(A) &= \int \mathrm{d}^4x A_\mu^T A_\mu^T \,, \\ &= \min \int \mathrm{d}^4x A_\mu A_\mu \\ &= A^2_{\mathrm{min}} \,, \end{aligned} \tag{6.3}$$

where

$$A_\mu = A_\mu^T + A_\mu^L \,, \tag{6.4}$$

and

$$\begin{aligned} A_\mu^T &= \left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right) A_\nu \\ A_\mu^L &= \frac{\partial_\mu\partial_\nu}{\partial^2} A_\nu \,, \end{aligned} \tag{6.5}$$

are the projections into the respective transverse and longitudinal components. In the Abelian case the gauge field A_μ achieves its minimum when $\partial_\mu A_\mu = 0$ and the longitudinal component $A_\mu^L = 0$. The transverse component A_μ^T is gauge invariant,

$$\delta A_\mu^T = 0 \,, \tag{6.6}$$

with

$$\delta A_\mu = -\partial_\mu \omega \,. \tag{6.7}$$

Note that due to the presence of the term $1/\partial^2$ in this operator it is non-local. A second invariant mass operator is the Stueckelberg term, [57],

$$\mathcal{O}_2(A) = \int \mathrm{d}^4x (A_\mu + \partial_\mu\phi)^2 \,, \tag{6.8}$$

where ϕ is a dimensionless scalar field and (6.8) is left invariant by the transformation

$$\delta A_\mu = -\partial_\mu\omega \,, \quad \delta\phi = \omega \,. \tag{6.9}$$

The operator $\mathcal{O}_2(A)$ can be seen to be identical to the operator $\mathcal{O}_1(A)$ by considering the equations of motion of the gauge invariant action

$$S = \int \mathrm{d}^4x \Big(\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{m^2}{2} (A_\mu + \partial_\mu \phi)^2\Big) , \tag{6.10}$$

where we see that

$$\partial_\mu A_\mu + \partial^2 \phi = 0 \quad \Rightarrow \quad \phi = -\frac{1}{\partial^2} \partial A . \tag{6.11}$$

A third Abelian mass operator is given by the non-local expression

$$\mathcal{O}_3(A) = -\frac{1}{2} \int \mathrm{d}^4x F_{\mu\nu} \frac{1}{\partial^2} F_{\mu\nu} . \tag{6.12}$$

By considering the explicit form of the Abelian field strength tensor it is possible to show that

$$\begin{aligned} \mathcal{O}_3(A) &= -\frac{1}{2} \int \mathrm{d}^4x (\partial_\mu A_\nu - \partial_\nu A_\mu) \frac{1}{\partial^2} (\partial_\mu A_\nu - \partial_\nu A_\mu) \\ &= \frac{1}{2} \int \mathrm{d}^4x \Big[A_\nu \frac{1}{\partial^2} (\partial^2 A_\nu - \partial_\mu \partial_\nu A_\mu) + A_\mu \frac{1}{\partial^2} (\partial^2 A_\mu - \partial_\nu \partial_\mu A_\nu) \Big] \\ &= \int \mathrm{d}^4x A_\nu \Big(A_\nu - \frac{\partial_\mu \partial_\nu}{\partial^2} A_\mu \Big) = \int \mathrm{d}^4x A_\mu^T A_\mu^T . \end{aligned} \tag{6.13}$$

Once again we see that in the Abelian case the non-local operator $\mathcal{O}_3(A)$ is equal to the operator $\mathcal{O}_1(A)$. The operator $\mathcal{O}_3(A)$ has the appealing characteristic that it may be cast into a local form through the introduction of suitable additional fields, [58],

$$-\frac{m^2}{4} \int \mathrm{d}^4x F_{\mu\nu} \frac{1}{\partial^2} F_{\mu\nu} \rightarrow \int \mathrm{d}^4x \Big(\frac{1}{4} \bar{B}_{\mu\nu} \partial^2 B_{\mu\nu} + \frac{im}{4} (F_{\mu\nu} B_{\mu\nu} - F_{\mu\nu} \bar{B}_{\mu\nu}) \Big) . \tag{6.14}$$

The localizing fields $\bar{B}_{\mu\nu}$ and $B_{\mu\nu}$ are complex and antisymmetric in their Lorentz indices and m is a mass parameter. The original form of the non-local operator $\mathcal{O}_3(A)$ is recovered by removing the additional fields using their equations of motion. The localized action (6.14) can be added to the usual QED Lagrangian without destroying renormalizability.

All of the Abelian mass operators considered above have the feature that they can be shown to be classically equivalent. Also, they can all be generalized to the case of non-Abelian gauge theory. As before, we begin by considering the operator A^2_{min}, and recall that the non-Abelian generalization is given by

$$\begin{aligned} A^2_{\text{min}} &\equiv \min_{\{u\}} \operatorname{tr} \int \mathrm{d}^4x A^u_\mu A^u_\mu \\ A^u_\mu &= u^\dagger A_\mu u + \frac{i}{g} u^\dagger \partial_\mu u , \end{aligned} \tag{6.15}$$

where gauge invariance derives from the minimization procedure along the gauge orbit of A_μ. The task of identifying an absolute minimum for the functional $\operatorname{tr} \int \mathrm{d}^4x A^u_\mu A^u_\mu$ provides an immediate indication of the difficulties associated with a generalization to non-Abelian gauge theory. The functional possesses many relative minima along a given gauge orbit. The difficulties associated with a suitable identification of a gauge invariant absolute minimum for the functional $\mathrm{tr} \int d^4x A^u_\mu A^u_\mu$ and absolute gauge fixing in Yang-Mills theory are closely related, [60],[61]. Identification of an absolute minimum for this functional derives from a restriction to the fundamental modular region.

The fundamental modular region is denoted by a functional space, contained within the Gribov region, where the possibility for the existence of Gribov copies is rigorously excluded. If we had an explicit characterization of the fundamental modular region practical for use with gauge theories at our disposal, then defining a gauge invariant, physically meaningful, gluon mass operator in QCD would, perhaps, follow naturally. In chapters 2 and 3, the more modest task of identifying the possibility of a Gribov copy using a perturbation around (deformation of) the Landau gauge and subsequent implementation of a restriction to the Gribov region suitable for use with Yang-Mills theory, which then must be fixed in the Landau gauge, was seen to be difficult enough. Although the definition of a Gribov copy, defined in terms of the horizon $\partial\Omega$, that was considered there is perfectly correct, it is well known that there are species of Gribov copies that can exist within the horizon $\partial\Omega$, [26]. A description of the fundamental modular region analogous to the description of Ω which implements the *global* Landau gauge fixing seen in the Gribov-Zwanziger action, is not available.

Now, it is well known that the functional space describing the operator A^2_{min} is restricted to that of the fundamental modular region, [60]. Restricting a consideration of the fundamental modular region to the more manageable task of identifying an absolute minimum for the functional

$$f_A[u] \quad = \quad \mathrm{tr} \int d^4x A^u_\mu A^u_\mu \ , \tag{6.16}$$

it is possible to make progress, [60]. An identification of relative minima for the functional $\mathrm{tr} \int d^4x A^u_\mu A^u_\mu$ confines the operator to the region contained within the Gribov horizon. The relative minima configurations of the functional occur when $u = h$ in (6.15) so that A^h_μ is a transverse field, $\partial_\mu A^h_\mu = 0$. The field configurations A^h_μ are expressed as a formal power series in the gauge field A_μ, [63], such that $h = h(A)$, leading to the expression

$$A^h_\mu \quad = \quad \Big(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\Big)\phi_\nu \ , \tag{6.17}$$

where

$$\phi_\nu \quad = \quad A_\nu - ig\Big[\frac{1}{\partial^2}\partial A, A_\nu\Big] + \frac{ig}{2}\Big[\frac{1}{\partial^2}\partial A, \partial_\nu\frac{1}{\partial^2}\partial A\Big] + O(A^3) \ . \tag{6.18}$$

In addition to expressing relative minima of the field configurations, an important property of the expansion A^h_μ is that it is gauge invariant since

$$\delta A^h_\mu = 0 \ \ , \ \ \delta A_\mu = -\partial_\mu\omega + ig[A_\mu, \omega] \ . \tag{6.19}$$

Given this, the explicit expression for $A^2_{\rm min}$ is given by

$$\begin{aligned} A^2_{\rm min} &= \operatorname{tr}\int d^4x A^h_\mu A^h_\mu \\ &= \frac{1}{2}\int d^4x\Big[A^a_\mu\Big(\delta_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{\partial^2}\Big)A^a_\nu - gf^{abc}\Big(\frac{\partial_\nu}{\partial^2}\partial A^a\Big)\Big(\frac{1}{\partial^2}\partial A^b\Big)A^c_\nu\Big] + O(A^4)\ , \end{aligned} \tag{6.20}$$

where we see that the operator is composed of an infinite number of non-local terms. In practice, the infinite non-local sum makes a generic choice of gauge fixing term for any Lagrangian including this gauge invariant operator impossible. In order to proceed it is necessary to adopt the Landau gauge condition, $\partial_\mu A^a_\mu = 0$, causing all non-local terms (the infinite sum) in (6.20) to disappear, leaving

$$A^2_{\rm min} = \frac{1}{2}\int d^4x A^a_\mu A^a_\mu \ \ ; \ \ \partial_\mu A^a_\mu = 0\ . \tag{6.21}$$

Indeed, the operator $A^2_{\rm min}$ has received a great deal of attention in the Landau gauge, see for example [39],[62], via the massive Yang-Mills action

$$S_m = \int d^4x\Big(\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \frac{m^2}{2}A^a_\mu A^a_\mu + b^a\partial_\mu A^a_\mu + \bar{c}^a\partial_\mu D^{ab}_\mu c^b\Big)\ , \tag{6.22}$$

where the Lagrange multiplier b^a enforces the Landau gauge condition. The massive Yang-Mills action (6.22) has been shown to be multiplicatively renormalizable to all orders in perturbation theory.

Since the aim is that of identifying a gauge invariant mass operator, for a treatment that is not restricted to the Landau gauge, we consider the non-Abelian generalization of the non-local operator

$$\int d^4x F_{\mu\nu}\frac{1}{\partial^2}F_{\mu\nu}\ . \tag{6.23}$$

This is done by replacing the space-time derivative ∂_μ by the covariant derivative D_μ such that,

$$\operatorname{tr}\int d^4x F_{\mu\nu}\frac{1}{D^2}F_{\mu\nu} \equiv \frac{1}{2}\int d^4x F^a_{\mu\nu}[(D^2)]^{-1}]^{ab}F^b_{\mu\nu}\ . \tag{6.24}$$

The operator (6.24) is seen to have a direct relationship to $A^2_{\rm min}$ via a direct route to a globally correct gauge fixing prescription for Euclidean Yang-Mills theory implementing a restriction of the functional space to a consideration of the fundamental modular region, rigorously free of Gribov copies, [60],[61]. In this study, the non-Abelian functional $A^2_{\rm min}$, (6.15), is expressed as a non local expansion in the field strength tensor

$$\begin{aligned} A^2_{\rm min} &= -\frac{1}{2}\operatorname{tr}\int d^4x\Big(F_{\mu\nu}\frac{1}{D^2}F_{\mu\nu} + 2i\frac{1}{D^2}F_{\lambda\mu}\Big[\frac{1}{D^2}D_\kappa F_{\kappa\lambda}, \frac{1}{D^2}D_\nu F_{\nu\mu}\Big] \\ &\quad - 2i\frac{1}{D^2}F_{\lambda\mu}\Big[\frac{1}{D^2}D_\kappa F_{\kappa\nu}, \frac{1}{D^2}D_\nu F_{\lambda\mu}\Big]\Big) + O(F^4)\ . \end{aligned} \tag{6.25}$$

Unlike the non-local Abelian operator $\int d^4x F_{\mu\nu}\frac{1}{\partial^2}F_{\mu\nu}$, the non-local expansion for the globally correct gauge fixing functional $A^2_{\rm min}$ and the operator (6.24) are not classically equivalent. The non-Abelian generalization coincides with the first term of the globally correct expansion. However, the non-Abelian expansion does contain the appealing feature that, each term in the expansion is individually gauge invariant. As in the Abelian case it is possible to localize (6.24) through the introduction of localizing fields. The localization procedure is not significantly more complicated than for the Abelian case, although, as we will see in the next chapter, identification of a suitable local action that is stable against radiative corrections (renormalizable) is a rather involved process. Given this, the operator (6.24) provides a manageable approximation to the non-Abelian functional A^2_{min}, it requires no special treatment of the fundamental modular or Gribov regions and can be added to the Yang-Mills Lagrangian without destroying renormalizability.

To show that the non-Abelian mass operator (6.15) is in fact given in terms of the expansion (6.25), we attempt a detailed consideration of the analysis given in [60], a rigorous mathematical proof of the statements made there will not be considered here but are detailed in, [61]. Determination of the absolute minimum achieved by the functional $\text{tr} \int d^4x A^u_\mu A^u_\mu$ is a highly non trivial task. As we have said, the functional possesses many relative minima along a given gauge orbit. A series of local minima are identified by restricting field configurations to the Gribov region. Field configurations confined to the Gribov region are expanded in a perturbative series and the absolute minimum is selected by expressing the transversality condition $\partial_\mu A_\mu = 0$ as a power series in the gluon field $h = h(A)$, [63]. From this we see clearly how the possibility of condensates with a mass dimension and the issue of gauge fixing in a non-Abelian theory, are closely related. The operator is considered using the properties of the functional $f_A[u]$, [60],[61],

$$\begin{aligned} f_A[u] &= \text{tr} \int d^4x A^u_\mu A^u_\mu \\ &= \text{tr} \int d^4x \left(u^\dagger A_\mu u + \frac{i}{g} u^\dagger \partial_\mu u \right) \left(u^\dagger A_\mu u + \frac{i}{g} u^\dagger \partial_\mu u \right) . \end{aligned} \tag{6.26}$$

For a given gauge field configuration A_μ, $f_A[u]$ is a functional defined on the gauge orbit of A_μ. Now, let $\mathcal{A}$ be the space of connections A^a_μ with finite Hilbert norm $||A||$, such that

$$||A||^2 = \text{tr} \int d^4x A_\mu A_\mu = \frac{1}{2} \int d^4x A^a_\mu A^a_\mu < +\infty \ , \tag{6.27}$$

and let $\mathcal{U}$ be the space of local gauge transformations u such that the Hilbert norm $||u^\dagger \partial u||$ is also finite

$$||u^\dagger \partial u||^2 = \text{tr} \int d^4x \left(u^\dagger \partial_\mu u \right) \left(u^\dagger \partial_\mu u \right) < +\infty \ . \tag{6.28}$$

For the functional to achieve an absolute minimum on the gauge orbit of A_μ, there must exist a field configuration, $h \in \mathcal{U}$, such that

$$\delta f_A[h] = 0 \tag{6.29}$$

$$\begin{aligned} \delta^2 f_A[h] &\geq 0 && (6.30)\\ f_A[h] &\leq f_A[u] \qquad \forall u \in \mathcal{U} \ . && (6.31) \end{aligned}$$

As such, the operator $A^2_{\rm min}$ will be given by

$$A^2_{\rm min} = \min_{\{u\}} \mathrm{tr} \int d^4x A^u_\mu A^u_\mu = f_A[h] \ . \qquad (6.32)$$

To evaluate $\delta f_A[h]$ and $\delta^2 f_A[h]$, set

$$\begin{aligned} u &= v = he^{ig\omega} \\ u^\dagger &= v^\dagger = e^{-ig\omega}h^\dagger \ , \end{aligned} \qquad (6.33)$$

where ω is an infinitesimal Hermitian matrix. Expanding to order ω^2, A^v_μ and $f_A[v]$ are evaluated in terms of ω and A^h_μ, the points on the gauge orbit of A_μ where the operator A^2 achieves a series of relative minima,

$$f_A[v] = f_A[h] + 2\mathrm{tr}\int d^4x \left(\omega\partial_\mu A^h_\mu\right) - \mathrm{tr}\int d^4x\omega\partial_\mu D_\mu(A^h)\omega + O(\omega^3) \ . \qquad (6.34)$$

Given this,

$$\begin{aligned} \delta f_A[h] &= 0 \Rightarrow \partial_\mu A^h_\mu = 0 && (6.35)\\ \delta^2 f_A[h] &> 0 \Rightarrow -\partial_\mu D_\mu(A^h) > 0 \ , && (6.36) \end{aligned}$$

defining a set of field configurations that give the relative minima of the functional $f_A[u]$ and are confined to the Gribov region Ω, which may be similarly defined

$$\Omega = \{A_\mu | \partial_\mu A_\mu = 0 \text{ and } -\partial_\mu D_\mu(A) > 0\} \ . \qquad (6.37)$$

By solving the transversality condition $\partial_\mu A^h_\mu = 0$ as a power series $h = h(A)$ in A_μ, it is possible to identify an absolute minimum gauge configuration. Starting with

$$A^h_\mu = h^\dagger A_\mu h + \frac{i}{g}h^\dagger\partial_\mu h \ , \qquad (6.38)$$

where

$$h = e^{ig\phi} = 1 + ig\phi - \frac{g^2}{2}\phi^2 + O(\phi^3) \ , \qquad (6.39)$$

then expanding A^h_μ to order ϕ^2 and imposing the transversality condition

$$\partial_\mu A^h_\mu = \partial_\mu\left(h^\dagger A_\mu h + \frac{i}{g}h^\dagger\partial_\mu h\right) = 0 \ . \qquad (6.40)$$

Following the method of [63], we then obtain an expression for $\partial^2\phi$ that can be solved iteratively for ϕ as a power series in A_μ. The absolute minimum gauge configuration A^h_μ is given by

$$\begin{aligned} A^h_\mu = {} & A_\mu - \frac{\partial_\mu}{\partial^2}\partial A + ig\left[A_\mu, \frac{1}{\partial^2}\partial A\right] + \frac{ig}{2}\left[\frac{1}{\partial^2}\partial A, \partial_\mu\frac{1}{\partial^2}\partial A\right] \\ & + ig\frac{\partial_\mu}{\partial^2}\left[\frac{\partial_\nu}{\partial^2}\partial A, A_\nu\right] + i\frac{g}{2}\frac{\partial_\mu}{\partial^2}\left[\frac{\partial A}{\partial^2}, \partial A\right] + O(A^3) \ , \end{aligned} \qquad (6.41)$$

and this is where we see the inherently non-local nature of the operator $A^2_{\rm min}$ begin to emerge. In order to demonstrate gauge invariance, A^h_μ is written using an equivalent expression

$$A^h_\mu = \left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)\phi_\nu \ , \tag{6.42}$$

where

$$\phi_\nu \ = \ A_\nu - ig\left[\frac{1}{\partial^2}\partial A, A_\nu\right] + \frac{ig}{2}\left[\frac{1}{\partial^2}\partial A, \partial_\nu\frac{1}{\partial^2}\partial A\right] + O(A^3) \ . \tag{6.43}$$

From this expression it is easy to see that A^h_μ is invariant under infinitesimal gauge transformations of the gauge fields in the right hand side of (6.41), order by order in the gauge coupling g

$$\delta A_\mu \ = \ -\partial_\mu\omega + ig[A_\mu, \omega] \ , \tag{6.44}$$

where

$$\delta\phi_\nu = -\partial_\nu\left(\omega - i\frac{g}{2}\left[\frac{\partial A}{\partial^2}, \omega\right]\right) + O(g^2) \ . \tag{6.45}$$

Returning now to question of Gribov copies and relative minima versus an absolute minimum, it is stipulated that by imposing the transversality condition on the perturbative solution for h, [60],

$$\partial_\mu A^h_\mu = \partial_\mu\left(h^\dagger A_\mu h \ + \ \frac{i}{g}h^\dagger\partial_\mu h\right) = 0 \ , \tag{6.46}$$

gives $h = 1$. That is, $A^h_\mu = A_\mu$, such that

$$A^2_{\rm min} = \min_{\{u\}} {\rm tr}\int d^4x A^u_\mu A^u_\mu = {\rm tr}\int d^4x A^h_\mu A^h_\mu \ . \tag{6.47}$$

This is clearly incorrect for any relative minima that might occur due to the existence of Gribov copies, since it only holds at the absolute minimum.

From the transversality condition it follows that

$$\begin{aligned} A^2_{\rm min} \ &= \ {\rm tr}\int d^4x A^h_\mu A^h_\mu \\ &= \ \frac{1}{2}\int d^4x\left[A^a_\mu\left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)A^a_\mu - gf^{abc}\left(\frac{\partial_\nu}{\partial^2}\partial A^a\right)\left(\frac{1}{\partial^2}\partial A^b\right)A^c_\nu\right] \\ &\quad + O(A^4) \ . \end{aligned} \tag{6.48}$$

Gauge invariance of the operator $A^2_{\rm min}$ makes it possible to express it as a series of gauge invariant terms that may be treated individually in the renormalization procedure. Rewriting $A^2_{\rm min}$ directly in terms of the Yang-Mills field strength tensor $F_{\mu\nu}$, gives

$$\begin{aligned} A^2_{\rm min} \ &= \ -\frac{1}{2}{\rm tr}\int d^4x\left(F_{\mu\nu}\frac{1}{D^2}F_{\mu\nu} + 2i\frac{1}{D^2}F_{\lambda\mu}\left[\frac{1}{D^2}D_\kappa F_{\kappa\lambda}, \frac{1}{D^2}D_\nu F_{\nu\mu}\right]\right. \\ &\quad \left. - \ 2i\frac{1}{D^2}F_{\lambda\mu}\left[\frac{1}{D^2}D_\kappa F_{\kappa\nu}, \frac{1}{D^2}D_\nu F_{\lambda\mu}\right]\right) + O(F^4) \ , \end{aligned} \tag{6.49}$$

leading to the possibility of considering the more manageable non-local operator

$$\mathcal{O} \ = \ {\rm tr}\int d^4x F_{\mu\nu}(D^2)^{-1}F_{\mu\nu} \ , \tag{6.50}$$

which is of dimension two. Incorporating a local realization of (6.50) into the QCD Lagrangian and establishing renormalizability using algebraic methods will be the subject of the next chapter.

Chapter 7

Localized gauge invariant mass operator

7.1 Localization and BRST invariance

We begin with addition of the non local mass operator S_O to the Yang-Mills action

$$
\begin{aligned}
S_{YM} &+ S_O \\
S_{YM} &= \frac{1}{4}\int \mathrm{d}^4x F^a_{\mu\nu}F^a_{\mu\nu} \\
S_O &= -\frac{m^2}{4}\int \mathrm{d}^4x F^a_{\mu\nu}[(D^2)^{-1}]^{ab}F^b_{\mu\nu} \ .
\end{aligned}
\tag{7.1}
$$

The non local action is made local using generalizations of the formulae

$$\int dydy^* \exp(-y^*Ay) = \det A \tag{7.2}$$

$$\int dydy^* \exp(-y^*Ay + \eta^*y + y^*\eta) = \det A \, \exp(\eta^* A^{-1}\eta) \ . \tag{7.3}$$

This process proceeds in a manner very similar to the localization procedure used in the Gribov-Zwanziger model, there are differences which we will mention briefly, after the localization fields have been introduced. This localization procedure introduces a pair of complex Bosonic antisymmetric tensor fields in the adjoint representation, $(B^a_{\mu\nu}, \bar{B}^a_{\mu\nu})$, leading to the expression, [58],

$$
\begin{aligned}
e^{-S_O} &= \int \mathcal{D}\bar{B}\mathcal{D}B(\det D^2)^6 \\
&\times \exp\left[-\left(\frac{1}{4}\int \mathrm{d}^4x\, \bar{B}^a_{\mu\nu}D^{ab}_\sigma D^{bc}_\sigma B^c_{\mu\nu} + \frac{im}{4}\int \mathrm{d}^4x\,(B-\bar{B})^a_{\mu\nu}F^a_{\mu\nu}\right)\right] \ .
\end{aligned}
\tag{7.4}
$$

The determinant, $(\det D^2)^6$, represents the Jacobian arising from the integration over the complex bosonic fields and may be accommodated in the exponent using anticommuting antisymmetric tensor field, $(\bar{G}^a_{\mu\nu}, G^a_{\mu\nu})$, via the expression

$$(\det D^2)^6 = \int \mathcal{D}\bar{G}\mathcal{D}G \exp\left(\frac{1}{4}\int d^4x\, \bar{G}^a_{\mu\nu} D^{ab}_\sigma D^{bc}_\sigma G^c_{\mu\nu}\right) \quad . \tag{7.5}$$

The combined *local* classical action is now given by

$$S_{YM} \;+\; S_{BG} \;+\; S_m \tag{7.6}$$

$$S_{BG} \;=\; \frac{1}{4}\int d^4x(\bar{B}^a_{\mu\nu} D^{ab}_\sigma D^{bc}_\sigma B^c_{\mu\nu} - \bar{G}^a_{\mu\nu} D^{ab}_\sigma D^{bc}_\sigma G^c_{\mu\nu}) \tag{7.7}$$

$$S_m \;=\; \frac{im}{4}\int d^4x(B - \bar{B})^a_{\mu\nu} F^a_{\mu\nu} \quad . \tag{7.8}$$

It is useful to look at the two sets of localizing fields together, for the Gribov-Zwanziger model, $\{\phi^{ab}_\mu, \bar{\phi}^{ab}_\mu, \omega^{ab}_\mu, \bar{\omega}^{ab}_\mu\}$, and for the non-local mass operator, $\{B^a_{\mu\nu}, \bar{B}^a_{\mu\nu}, G^a_{\mu\nu}, \bar{G}^a_{\mu\nu}\}$. We immediately see that the major difference is in the Lorentz and colour indices. The double indices for all localizing fields are antisymmetric.

The gauge invariance of the original non-local action is not destroyed by the localization procedure. Indeed, it is straightforward to check that (7.6) is left invariant by the gauge transformations

$$\begin{array}{llll} \delta A^a_\mu & = -D^{ab}_\mu \omega^b & \delta B^a_{\mu\nu} & = \; g f^{abc}\omega^b B^c_{\mu\nu} \\ \delta \bar{B}^a_{\mu\nu} & = g f^{abc}\omega^b \bar{B}^c_{\mu\nu} & \delta G^a_{\mu\nu} & = \; g f^{abc}\omega^b G^c_{\mu\nu} \\ \delta \bar{G}^a_{\mu\nu} & = g f^{abc}\omega^b \bar{G}^c_{\mu\nu} \; , & & \end{array} \tag{7.9}$$

leading to the desired result

$$\delta(S_{YM} + S_{BG} + S_m) = 0 \quad . \tag{7.10}$$

Having identified the local classical action the next task is to find a multiplicatively renormalizable quantized action. The field operators $B^a_{\mu\nu}F^a_{\mu\nu}$ and $\bar{B}^a_{\mu\nu}F^a_{\mu\nu}$ of (7.8) are treated as composite operators coupled to external sources $(V_{\sigma\rho\mu\nu}, \bar{V}_{\sigma\rho\mu\nu})$ as in the Gribov- Zwanziger model, [30]. This process is also considered in detail in [32]. This amounts to splitting the action (7.6) into two and considering the composite operators as an insertion,

$$\begin{aligned} S &= S_{YM} + S_{BG} + S_m \\ &= (S_{YM} + S_{BG}) + S_{\text{ext}} \end{aligned} \tag{7.11}$$

$$S_{\text{ext}} \;=\; \frac{1}{4}\int d^4x(V_{\sigma\rho\mu\nu}\bar{B}^a_{\sigma\rho}F^a_{\mu\nu} - \bar{V}_{\sigma\rho\mu\nu}B^a_{\sigma\rho}F^a_{\mu\nu}) \quad . \tag{7.12}$$

The original mass action (7.8) is recovered by identifying the sources $(V_{\sigma\rho\mu\nu}, \bar{V}_{\sigma\rho\mu\nu})$ with a physical value

$$V_{\sigma\rho\mu\nu}|_{\text{phys}} = \bar{V}_{\sigma\rho\mu\nu}|_{\text{phys}} = -\frac{im}{2}(\delta_{\sigma\mu}\delta_{\rho\nu} - \delta_{\sigma\nu}\delta_{\rho\nu}) \quad . \tag{7.13}$$

Separating the local action into two pieces it is possible to study the Green function renormalization properties of the massless action $(S_{YM}+S_{BG})$ and insert the composite operators $(B^a_{\mu\nu}F^a_{\mu\nu}, \bar{B}^a_{\mu\nu}F^a_{\mu\nu})$ into a massless theory. The symmetry content of the massless theory will be essential for establishing renormalizability. It is considered by gauge fixing in an arbitrary linear covariant gauge, using the Faddeev-Popov method, and exploiting the resulting BRST invariance,

$$\begin{aligned} S &= S_{YM}+S_{BG}+S_{gf} &(7.14)\\ S_{gf} &= \int \mathrm{d}^4x \left(\frac{\alpha}{2}b^a b^a + b^a\partial_\mu A^a_\mu + \bar{c}^a\partial_\mu D^{ab}_\mu c^b\right) \ . &(7.15)\end{aligned}$$

In the gauge fixed theory the transformations (7.9) are replaced by the nilpotent BRST transformations,

$$\begin{aligned} sA^a_\mu &= -D^{ab}_\mu c^b & sc^a &= \frac{g}{2}f^{abc}c^a c^b\\ sB^a_{\mu\nu} &= gf^{abc}c^b B^c_{\mu\nu} + G^a_{\mu\nu} & s\bar{B}^a_{\mu\nu} &= gf^{abc}c^b\bar{B}^c_{\mu\nu}\\ sG^a_{\mu\nu} &= gf^{abc}c^b G^c_{\mu\nu} & s\bar{G}^a_{\mu\nu} &= gf^{abc}c^b\bar{G}^c_{\mu\nu} + \bar{B}^a_{\mu\nu}\\ s\bar{c}^a &= b^a & sb^a &= 0\\ s^2 &= 0 \ , \end{aligned} \qquad (7.16)$$

leaving the gauge fixed action BRST invariant

$$s(S_{YM}+S_{BG}+S_{gf}) = 0 \ . \qquad (7.17)$$

Using the transformations (7.16) to express S_{BG} and S_{gf} as pure BRST variations and remembering that these operators are nilpotent, $s^2=0$,

$$\begin{aligned} S_{BG} &= \frac{1}{4}s\int \mathrm{d}^4x \bar{G}^a_{\mu\nu}D^{ab}_\sigma D^{bc}_\sigma B^c_{\mu\nu}\\ S_{gf} &= s\int \mathrm{d}^4x\left(\frac{\alpha}{2}\bar{c}^a b^a + \bar{c}^a\partial_\mu A^a_\mu\right) \ , \end{aligned} \qquad (7.18)$$

it is easy to see the BRST invariance of the gauge fixed action. In analogy with the Gribov-Zwanziger model, the non-local action considered here also displays a global, $U(f)$, invariance. Now, the global invariance is used to define the composite index, $i \equiv \{\mu,\nu\}$, by setting

$$\{B^a_i, \bar{B}^a_i, G^a_i, \bar{G}^a_i\} = \frac{1}{2}\{B^a_{\mu\nu}, \bar{B}^a_{\mu\nu}, G^a_{\mu\nu}, \bar{G}^a_{\mu\nu}\} \ , \qquad (7.19)$$

where because these fields are antisymmetric in their Lorentz indices, in four dimensions, $f=1,\cdots,6$. Noting this,

$$S_{BG} = \int \mathrm{d}^4x(\bar{B}^a_i D^{ab}_\mu D^{bc}_\mu B^c_i - \bar{G}^a_i D^{ab}_\mu D^{bc}_\mu G^c_i) \ , \qquad (7.20)$$

and invariance is expressed by

$$Q_{ij}S = 0 \ , \qquad (7.21)$$

where

$$Q_{ij} = \int \mathrm{d}^4x \left(B^a_i \frac{\delta}{\delta B^a_j} - \bar{B}^a_j \frac{\delta}{\delta \bar{B}^a_i} + G^a_i \frac{\delta}{\delta G^a_j} - \bar{G}^a_j \frac{\delta}{\delta \bar{G}^a_i} \right) \quad . \tag{7.22}$$

As in the Gribov-Zwanziger model, it is possible to define a new quantum number, $Q_f = Q_{ii}$, where the Q_f charge for each field, together with the Faddeev-Popov ghost number and dimension of each field is detailed in Table 7.1.

	A^a_μ	c^a	$\bar{c}^a$	b^a	B^a_i	$\bar{B}^a_i$	G^a_i	$\bar{G}^a_i$
dimension	1	0	2	2	1	1	1	1
ghost number	0	1	-1	0	0	0	1	-1
Q_f charge	0	0	0	0	1	-1	1	1

Table 7.1: Quantum numbers for local operator parameters.

In order to proceed, it is necessary to introduce the source terms, (7.12), in a BRST invariant way. This is done using the source term

$$\begin{aligned} S_{\mathrm{aux}} \;&=\; s \int \mathrm{d}^4x \Big[\left(V_{i\mu\nu} \bar{G}^a_i - \bar{U}_{i\mu\nu} B^a_i \right) F^a_{\mu\nu} + \chi_1 \bar{U}_{i\mu\nu} \partial^2 V_{i\mu\nu} + \chi_2 \bar{U}_{i\mu\nu} \partial_\mu \partial_\alpha V_{i\nu\alpha} \\ &\qquad - \zeta \left(\bar{U}_{i\mu\nu} V_{i\mu\nu} \bar{V}_{j\alpha\beta} V_{j\alpha\beta} - \bar{U}_{i\mu\nu} V_{i\mu\nu} \bar{U}_{j\alpha\beta} U_{j\alpha\beta} \right) \Big] \quad , \end{aligned} \tag{7.23}$$

with

$$\begin{aligned} sV_{i\mu\nu} &= U_{i\mu\nu} \qquad & sU_{i\mu\nu} &= 0 \\ s\bar{U}_{i\mu\nu} &= \bar{V}_{i\mu\nu} \qquad & s\bar{V}_{i\mu\nu} &= 0 \quad . \end{aligned} \tag{7.24}$$

The parameters, χ_1, χ_2 and ζ are left free for renormalization purposes. When the sources attain their physical values, on distinct indices, $U_{\alpha\beta\mu\nu} = \bar{U}_{\alpha\beta\mu\nu} = 0$, and

$$S_{\mathrm{aux}}\big|_{\mathrm{phys}} \to S_m - \frac{9}{4} \int \mathrm{d}^4x \zeta m^4 \quad , \tag{7.25}$$

the original mass term is recovered along with a constant factor that does not affect the behaviour of the model.

Finally, it is necessary to introduce external sources for all quantities with variations that are non linear in the fields. That is, products of fields $(A^a_\mu, c^a, B^a_i, \bar{B}^a_i, G^a_i, \bar{G}^a_i)$ in the BRST transformations (7.16) must be treated as composite operators and therefore, in the quantum theory, require additional external sources. The additional field source terms are introduced into the complete action expressed in terms of a pure nilpotent BRST variation using,

$$S_{\mathrm{ext}} \;=\; s \int \mathrm{d}^4x \left(-\Omega^a_\mu A^a_\mu + L^a c^a - \bar{Y}^a_i B^a_i - Y^a_i \bar{B}^a_i + \bar{X}^a_i G^a_i + X^a_i \bar{G}^a_i \right) \; , \tag{7.26}$$

The complete action Σ, gauge fixed in an arbitrary linear covariant gauge, is given by

$$\begin{aligned}
\Sigma &= S_{YM} + S_{gf} + S_{BG} + S_{\text{aux}} + S_{\text{ext}} \\
&= S_{YM} + \int \mathrm{d}^4x \Big[\frac{\alpha}{2} b^a b^a + b^a \partial_\mu A^a_\mu + \bar{c}^a \partial_\mu D^{ab}_\mu c^b + \bar{B}^a_i D^{ab}_\mu D^{bc}_\mu B^c_i - \bar{G}^a_i D^{ab}_\mu D^{bc}_\mu G^c_i \\
&\quad + (\bar{U}_{i\mu\nu} G^a_i + V_{i\mu\nu} \bar{B}^a_i - \bar{V}_{i\mu\nu} B^a_i + U_{i\mu\nu} \bar{G}^a_i) F^a_{\mu\nu} \\
&\quad + \chi_1 (\bar{V}_{i\mu\nu} \partial^2 V_{i\mu\nu} - \bar{U}_{i\mu\nu} \partial^2 U_{i\mu\nu}) \\
&\quad + \chi_2 (\bar{V}_{i\mu\nu} \partial_\mu \partial_\alpha V_{i\mu\nu} - \bar{U}_{i\mu\nu} \partial_\mu \partial_\alpha U_{i\mu\nu}) \\
&\quad + \zeta (\bar{U}_{i\mu\nu} U_{i\mu\nu} \bar{U}_{j\alpha\beta} U_{j\alpha\beta} + \bar{V}_{i\mu\nu} V_{i\mu\nu} \bar{V}_{j\alpha\beta} V_{j\alpha\beta} - 2 \bar{U}_{i\mu\nu} U_{i\mu\nu} \bar{V}_{j\alpha\beta} V_{j\alpha\beta}) \\
&\quad - \Omega^a_\mu D^{ab}_\mu c^b + \frac{g}{2} f^{abc} L^a c^b c^c \\
&\quad + g f^{abc} (\bar{Y}^a_i c^b B^c_i + Y^a_i c^b \bar{B}^c_i + \bar{X}^a_i c^b G^c_i + X^a_i c^b \bar{G}^c_i) \Big] \ .
\end{aligned} \tag{7.27}$$

7.2 Algebraic renormalization procedure

The task of producing a localized, gauge fixed and BRST invariant action for the non-local mass operator $S_{\mathcal{O}}$, described in the previous section proceeded in a manner similar to that for the Gribov-Zwanziger action of chapter 3. By definition, the Gribov-Zwanziger (GZ) action was fixed in the Landau gauge, the action considered above is fixed in an arbitrary linear covariant gauge. The Landau gauge displays finiteness properties not observed in generic linear or non-linear gauge fixing, [65],[66]. Given this, it should be expected that applying the algebraic renormalization procedure to this model will not be as straightforward as the algebraic analysis of chapter 3, [34]. We recall from chapter 3 that, the algebraic renormalization procedure tests a model's stability with respect to radiative corrections by identifying the most general counterterm Σ_Δ which satisfies all of the Ward identities of the classical action. The object Σ_Δ is an integrated local polynomial in the *classical* fields and sources and, like the starting action, has dimension bounded by four with vanishing ghost number and Q_f charge.

Because the model has been constructed in a manifestly BRST invariant way, it displays a rich symmetry content, that is, Σ satisfies a number of Ward identities with which Σ_Δ must also be compatible, restricting the most general possibilities for Σ_Δ, [64],[58]. Perhaps the best way to introduce the additional complications associated with this model, generic gauge fixing, is by showing the full detail for the counterterm, it was found to be, [58],

$$\begin{aligned}
\Sigma_\Delta &= a_0 S_{YM} + a_1 \int \mathrm{d}^4x A^a_\mu \frac{\delta S_{YM}}{\delta A^a_\mu} \\
&\quad + \int \mathrm{d}^4x \Big((a_1 + a_2)(\Omega^a_\mu + \partial_\mu \bar{c}^a) \partial_\mu c^a + a_2 g f^{abc} (\Omega^a_\mu + \partial_\mu \bar{c}^a) A^b_\mu c^c \\
&\quad - a_2 \frac{g}{2} f^{abc} L^a c^b c^c + (2a_3 + a_4) \bar{B}^a_i \partial^2 B^a_i - (2a_3 + a_4) \bar{G}^a_i \partial^2 G^a_i
\end{aligned}$$

$$
\begin{aligned}
&- (a_1 + 2a_3 + a_4) g f^{abc} \bar{B}_i^a (\partial_\mu A_\mu^b + 2A_\mu^b \partial_\mu) B_i^c \\
&+ (2a_1 + 2a_3 + a_4) g^2 f^{abd} f^{bce} \bar{B}_i^a A_\mu^d A_\mu^e B_i^c \\
&+ (a_1 + 2a_3 + a_4) g f^{abc} \bar{G}_i^a (\partial_\mu A_\mu^b + 2A_\mu^b \partial_\mu) G_i^c \\
&- (2a_1 + 2a_3 + a_4) g^2 f^{abd} f^{bce} \bar{G}_i^a A_\mu^d A_\mu^e G_i^c \\
&- a_2 g f^{abc} c^a (\bar{Y}_i^b B_i^c + Y_i^b \bar{B}_i^c - \bar{X}_i^b G_i^c - X_i^b \bar{G}_i^c) \\
&+ \left[(a_1 + a_3 + a_5) 2\partial_\mu A_\nu^a + (2a_1 + a_3 + a_5) g f^{abc} A_\mu^b A_\nu^c \right] \\
&\quad \times (\bar{U}_{i\mu\nu} G_i^a + V_{i\mu\nu} \bar{B}_i^a + U_{i\mu\nu} \bar{G}_i^a - \bar{V}_{i\mu\nu} B_i^a) \\
&+ \frac{\lambda^{abcd}}{16} (\bar{B}_i^a B_i^b - \bar{G}_i^a G_i^b)(\bar{B}_i^c B_i^d - \bar{G}_i^c G_i^d) \\
&+ a_7 (\bar{B}_i^a B_i^a - \bar{G}_i^a G_i^a)(\bar{V}_{i\mu\nu} V_{i\mu\nu} - \bar{U}_{i\mu\nu} U_{i\mu\nu}) \\
&+ a_8 \Big(\bar{B}_i^a G_j^a V_{i\mu\nu} \bar{U}_{j\mu\nu} + \bar{G}_i^a G_j^a U_{i\mu\nu} \bar{U}_{j\mu\nu} + \bar{B}_i^a B_j^a V_{i\mu\nu} \bar{V}_{j\mu\nu} \\
&- \bar{G}_i^a B_j^a U_{i\mu\nu} \bar{V}_{j\mu\nu} - G_i^a B_j^a \bar{U}_{i\mu\nu} \bar{V}_{j\mu\nu} + \bar{G}_i^a \bar{B}_j^a U_{i\mu\nu} V_{j\mu\nu} \\
&- \frac{1}{2} B_i^a B_j^a \bar{V}_{i\mu\nu} \bar{V}_{j\mu\nu} + \frac{1}{2} G_i^a G_j^a \bar{U}_{i\mu\nu} \bar{U}_{j\mu\nu} - \frac{1}{2} \bar{B}_i^a \bar{B}_j^a V_{i\mu\nu} V_{j\mu\nu} \\
&+ \frac{1}{2} \bar{G}_i^a \bar{G}_j^a U_{i\mu\nu} U_{j\mu\nu} \Big) + a_9 \zeta (\bar{V}_{i\mu\nu} V_{i\mu\nu} - \bar{U}_{i\mu\nu} U_{i\mu\nu})^2 \\
&+ a_{10} \chi_1 (\bar{V}_{i\mu\nu} \partial^2 V_{i\mu\nu} - \bar{U}_{i\mu\nu} \partial^2 U_{i\mu\nu}) \\
&+ a_{11} \chi_1 (\bar{V}_{i\mu\nu} \partial_\mu \partial_\alpha V_{i\nu\alpha} - \bar{U}_{i\mu\nu} \partial_\mu \partial_\alpha U_{i\nu\alpha})) \quad , \qquad (7.28)
\end{aligned}
$$

where a_i, $i = 1, \cdots, 11$, represent as yet undetermined constants. Unlike the most general counterterm for the Landau gauge GZ model, Σ_Δ is now given by an expression which includes a term with a new invariant rank four tensor coupling λ^{abcd}. For a starting action that is stable against radiative corrections, the parameters a_i correspond to a multiplicative renormalization of the fields, couplings and sources of the starting classical action Σ. In particular, the term

$$
\begin{aligned}
\Sigma_\lambda \;=\; & \int d^4x \Bigg(\frac{\lambda^{abcd}}{16} (\bar{B}_i^a B_i^b - \bar{G}_i^a G_i^b)(\bar{B}_i^c B_i^d - \bar{G}_i^c G_i^d) \\
&+ a_7 (\bar{B}_i^a B_i^a - \bar{G}_i^a G_i^a)(\bar{V}_{i\mu\nu} V_{i\mu\nu} - \bar{U}_{i\mu\nu} U_{i\mu\nu}) \\
&+ a_8 \Big(\bar{B}_i^a G_j^a V_{i\mu\nu} \bar{U}_{j\mu\nu} + \bar{G}_i^a G_j^a U_{i\mu\nu} \bar{U}_{j\mu\nu} + \bar{B}_i^a B_j^a V_{i\mu\nu} \bar{V}_{j\mu\nu} - \bar{G}_i^a B_j^a U_{i\mu\nu} \bar{V}_{j\mu\nu} \\
&- G_i^a B_j^a \bar{U}_{i\mu\nu} \bar{V}_{j\mu\nu} + \bar{G}_i^a \bar{B}_j^a U_{i\mu\nu} V_{j\mu\nu} - \frac{1}{2} B_i^a B_j^a \bar{V}_{i\mu\nu} \bar{V}_{j\mu\nu} + \frac{1}{2} G_i^a G_j^a \bar{U}_{i\mu\nu} \bar{U}_{j\mu\nu} \\
&- \frac{1}{2} \bar{B}_i^a \bar{B}_j^a V_{i\mu\nu} V_{j\mu\nu} + \frac{1}{2} \bar{G}_i^a \bar{G}_j^a U_{i\mu\nu} U_{j\mu\nu} \Big) \Bigg) \quad , \qquad (7.29)
\end{aligned}
$$

in (7.28), which is compatible with all of the Ward identities satisfied by the starting action Σ, cannot be reabsorbed through a renormalization of the fields and couplings of that action. The algebraic renormalization procedure has shown that Σ, (7.27), is not stable against radiative corrections and is therefore not the most general local invariant action compatible with the Ward identities, [58]. In order to have a starting action that is stable against radiative corrections, it is necessary to include an additional term

$$
\begin{aligned}
S_\lambda &= \int \mathrm{d}^4x \left(\frac{\lambda^{abcd}}{16}(\bar{B}_i^a B_i^b - \bar{G}_i^a G_i^b)(\bar{B}_i^c B_i^d - \bar{G}_i^c G_i^d) \right. \\
&\quad + \lambda_1 (\bar{B}_i^a B_i^a - \bar{G}_i^a G_i^a)(\bar{V}_{i\mu\nu} V_{i\mu\nu} - \bar{U}_{i\mu\nu} U_{i\mu\nu}) \\
&\quad + \lambda_3 \Big(\bar{B}_i^a G_j^a V_{i\mu\nu} \bar{U}_{j\mu\nu} + \bar{G}_i^a G_j^a U_{i\mu\nu} \bar{U}_{j\mu\nu} + \bar{B}_i^a B_j^a V_{i\mu\nu} \bar{V}_{j\mu\nu} - \bar{G}_i^a B_j^a U_{i\mu\nu} \bar{V}_{j\mu\nu} \\
&\quad - G_i^a B_j^a \bar{U}_{i\mu\nu} \bar{V}_{j\mu\nu} + \bar{G}_i^a \bar{B}_j^a U_{i\mu\nu} V_{j\mu\nu} - \frac{1}{2} B_i^a B_j^a \bar{V}_{i\mu\nu} \bar{V}_{j\mu\nu} + \frac{1}{2} G_i^a G_j^a \bar{U}_{i\mu\nu} \bar{U}_{j\mu\nu} \\
&\quad \left. - \frac{1}{2} \bar{B}_i^a \bar{B}_j^a V_{i\mu\nu} V_{j\mu\nu} + \frac{1}{2} \bar{G}_i^a \bar{G}_j^a U_{i\mu\nu} U_{j\mu\nu} \Big) \right) \quad , \qquad (7.30)
\end{aligned}
$$

and carry out the algebraic renormalization procedure again using the starting action

$$
\bar{\Sigma} = S_{YM} + S_{gf} + S_{BG} + S_{\rm aux} + S_{\rm ext} + S_\lambda \ . \qquad (7.31)
$$

Using $\bar{\Sigma}$ in the algebraic renormalization analysis alters the form taken by the Ward identities, in particular the Slavnov-Taylor identity. Repeating the algebraic analysis for the action $\bar{\Sigma}$ produces a second most general allowed counterterm $\bar{\Sigma}_\Delta$ which can be reabsorbed by renormalization of the parameters of the starting action, [58]. After it is established that $\bar{\Sigma}$ is stable against radiative corrections, it is shown that when the sources, $(V_{i\mu\nu}, \bar{V}_{i\mu\nu}, U_{i\mu\nu}, \bar{U}_{i\mu\nu})$, attain their physical value,

$$
\begin{aligned}
S_\lambda|_{\rm phys} &= \int \mathrm{d}^4x \left[-\frac{3\lambda_1 m^2}{8}(\bar{B}_{\mu\nu}^a B_{\mu\nu}^a - \bar{G}_{\mu\nu}^a G_{\mu\nu}^a) + \frac{\lambda_3 m^2}{32}(\bar{B}_{\mu\nu}^a - B_{\mu\nu}^a)^2 \right. \\
&\quad \left. + \frac{\lambda^{abcd}}{16}(\bar{B}_{\mu\nu}^a B_{\mu\nu}^b - \bar{G}_{\mu\nu}^a G_{\mu\nu}^b)(\bar{B}_{\rho\sigma}^c B_{\rho\sigma}^d - \bar{G}_{\rho\sigma}^c G_{\rho\sigma}^d) \right] \ . \qquad (7.32)
\end{aligned}
$$

Also, the algebraic renormalization procedure dictates that the renormalization constants for the localizing fields, $\{\bar{B}_{\mu\nu}^a, B_{\mu\nu}^a, \bar{G}_{\mu\nu}^a, G_{\mu\nu}^a\}$, are all identical, that is, [58],

$$
Z_{\bar{B}} = Z_B = Z_{\bar{G}} = Z_G \ . \qquad (7.33)
$$

Finally, the local renormalizable action, where the field sources have been set equal to zero, is given by

$$
\begin{aligned}
S &= S_{YM} + S_{BG} + S_m + S_\lambda|_{\rm phys} + S_{gf} \\
&= \int \mathrm{d}^4x \left[\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{im}{4}(B - \bar{B})_{\mu\nu}^a F_{\mu\nu}^a \right. \\
&\quad + \frac{1}{4}(\bar{B}_{\mu\nu}^a D_\sigma^{ab} D_\sigma^{bc} B_{\mu\nu}^c - \bar{G}_{\mu\nu}^a D_\sigma^{ab} D_\sigma^{bc} G_{\mu\nu}^c) \\
&\quad + \left(\frac{\alpha}{2} b^a b^a + b^a \partial_\mu A_\mu^a + \bar{c}^a \partial_\mu D_\mu^{ab} c^b \right) \ . \\
&\quad - \frac{3\lambda_1 m^2}{8}(\bar{B}_{\mu\nu}^a B_{\mu\nu}^a - \bar{G}_{\mu\nu}^a G_{\mu\nu}^a) + \frac{\lambda_3 m^2}{32}(\bar{B}_{\mu\nu}^a - B_{\mu\nu}^a)^2 \\
&\quad \left. + \frac{\lambda^{abcd}}{16}(\bar{B}_{\mu\nu}^a B_{\mu\nu}^b - \bar{G}_{\mu\nu}^a G_{\mu\nu}^b)(\bar{B}_{\rho\sigma}^c B_{\rho\sigma}^d - \bar{G}_{\rho\sigma}^c G_{\rho\sigma}^d) \right] \ . \qquad (7.34)
\end{aligned}
$$

As in the case of the Gribov-Zwanziger model, since (7.34) has been derived in a manifestly BRST invariant way, the renormalization properties of the ordinary QCD objects are unaffected and the usual properties of QCD, such as asymptotic freedom, remain intact. The task of the next chapter will be to use (7.34) to do loop calculations in the $\overline{\text{MS}}$ scheme. The main focus will be that of establishing the three loop anomalous dimension for the mass operator. Also we will establish that the usual QCD renormalization is unaltered at three loops and furnish the new constants, $\tilde{a}_i$, for the localizing fields, $\{B^a_{\mu\nu}, \bar{B}^a_{\mu\nu}, G^a_{\mu\nu}, \bar{G}^a_{\mu\nu}\}$, derived in the second algebraic renormalization with their explicit three loop $\overline{\text{MS}}$ values.

Chapter 8

Three Loop Calculation

8.1 Operator Insertion

We now come to the main new calculation in this thesis, determination of the anomalous dimension, to three loop order in the $\overline{\text{MS}}$ scheme, for the gauge invariant mass operator

$$\mathcal{O} = \left(B^a_{\mu\nu} - \bar{B}^a_{\mu\nu} \right) F^a_{\mu\nu} \ . \tag{8.1}$$

The calculation extends the one and two loop results presented in [58] and [59] respectively, and proceeds by making reference to the specific Lagrangian

$$\begin{aligned} L &= -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} - \frac{1}{2\alpha} (\partial_\mu A^a_\mu)^2 - \bar{c}^a \partial_\mu D_\mu c^a + i \bar{\psi}^{iI} \not{D} \psi^{iI} \\ &+ \frac{1}{4} \left(\bar{B}^a_{\mu\nu} D^{ab}_\sigma D^{bc}_\sigma B^c_{\mu\nu} - \bar{G}^a_{\mu\nu} D^{ab}_\sigma D^{bc}_\sigma G^c_{\mu\nu} \right) + \frac{im}{4} \left(B^a_{\mu\nu} - \bar{B}^a_{\mu\nu} \right) F^a_{\mu\nu} \\ &+ \frac{1}{16} \lambda^{abcd} \left(\bar{B}^a_{\mu\nu} B^b_{\mu\nu} - \bar{G}^a_{\mu\nu} G^b_{\mu\nu} \right) \left(\bar{B}^c_{\sigma\rho} B^d_{\sigma\rho} - \bar{G}^c_{\sigma\rho} G^d_{\sigma\rho} \right) , \end{aligned} \tag{8.2}$$

this Lagrangian stems from the action, (7.34), presented at the end of chapter 7, where here, the Lagrange multiplier b^a, has been removed, using its equation of motion, and the independent mass operators, λ_1 and λ_2, have been set equal to zero. In analogy with the algebraic renormalization procedures outlined in chapters 3 and 7, and in distinction to the loop analysis of chapters 4 and 5, by considering the case when m is equal to zero it is possible to regard (8.2) to be a massless Lagrangian and treat the mass operator separately as an insertion. The massless Lagrangian describes a gauge theory, fixed in an arbitrary linear covariant gauge, which in addition to the usual gluon, quark and Faddeev-Popov ghost fields, contains the auxiliary fields, $\{B^a_{\mu\nu}, \bar{B}^a_{\mu\nu}, G^a_{\mu\nu}, \bar{G}^a_{\mu\nu}\}$, and a tensor coupling, λ^{abcd}. The auxiliary fields are antisymmetric in their Lorentz indices $\{\mu\nu\}$. By treating the mass operator as a perturbation it is possible to extract the anomalous dimension of $\mathcal{O}$ using massless fields. Recalling the Lorentz symmetry properties of the auxiliary fields, it is straightforward to derive the propagators from the massless Lagrangian, such that

$$
\begin{aligned}
\langle A^a_\mu(p) A^b_\nu(-p) \rangle &= -\frac{\delta^{ab}}{p^2}\left[\delta_{\mu\nu} - (1-\alpha)\frac{p_\mu p_\nu}{p^2}\right] \\
\langle c^a(p) \bar{c}^b(-p) \rangle &= \frac{\delta^{ab}}{p^2} \\
\langle \psi(p) \bar{\psi}(-p) \rangle &= \frac{\not{p}}{p^2} \\
\langle B^a_{\mu\nu}(p) \bar{B}^b_{\sigma\rho}(-p) \rangle &= -\frac{\delta^{ab}}{2p^2}\left[\delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\sigma}\right] \\
\langle G^a_{\mu\nu}(p) \bar{G}^b_{\sigma\rho}(-p) \rangle &= -\frac{\delta^{ab}}{2p^2}\left[\delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\sigma}\right] \ .
\end{aligned} \tag{8.3}
$$

Turning to the explicit forms taken by the operator

$$
\begin{aligned}
\mathcal{O} &= \left(B^a_{\mu\nu} - \bar{B}^a_{\mu\nu}\right) F^a_{\mu\nu} \\
&= \left(B^a_{\mu\nu} - \bar{B}^a_{\mu\nu}\right)\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu\right) \\
&= -\left(B^a_{\mu\nu} - \bar{B}^a_{\mu\nu}\right)\left(2\partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu\right) \ ,
\end{aligned} \tag{8.4}
$$

we choose to investigate the mass anomalous dimension using a modified 2-point function derived using the first term in (8.4). This is done in momentum space according to the prescription,

$$
\int \mathrm{d}^d x \mathcal{L}(x) = \int \mathrm{d}^d p \mathcal{L}(p) \ , \tag{8.5}
$$

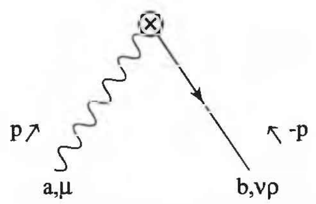
where

$$
\phi(x) = \int_p e^{-ipx}\tilde{\phi}(p) \ . \tag{8.6}
$$

Given this

$$
\begin{aligned}
-2\int_x B^a_{\mu\nu}(x)\partial_\nu A^a_\mu(x) &= -2\delta^{ab}\int_x\int_p\int_q \left(B^b_{\mu\nu}(q)\partial_\nu A^a_\mu(p)\right) e^{-i(p+q)x} \\
&= -2\delta^{ab}\int_x\int_p\int_q \left(B^b_{\mu\nu}(q)(-ip_\nu)A^a_\mu(p)\right) e^{-i(p+q)x} \\
&= -\delta^{ab}\int_x\int_p\int_q \left[\delta_{\mu\rho}B^b_{\rho\nu}(q)\partial_\nu A^a_\mu(p) - \delta_{\mu\nu}B^b_{\rho\nu}(q)\partial_\rho A^a_\mu(p)\right] \\
&\qquad \times e^{-i(p+q)x} \\
&= -\delta^{ab}\int_x\int_p\int_q \\
&\qquad \times \left[\delta_{\mu\rho}B^b_{\rho\nu}(q)(-ip_\nu)A^a_\mu(p) - \delta_{\mu\nu}B^b_{\rho\nu}(q)(-ip_\rho)A^a_\mu(p)\right] \\
&\qquad \times e^{-i(p+q)x} \\
&= \int_p A^a_\mu(p)\left[i\delta^{ab}\left(\delta_{\mu\nu}p_\rho - \delta_{\mu\rho}p_\nu\right)\right] B^b_{\nu\rho}(-p) \ .
\end{aligned} \tag{8.7}
$$

As such, we calculate the mass anomalous dimension using a modified 2-point function with the Feynman rule



$$
\begin{aligned}
&= \langle A^a_\mu(p) \mathcal{O} B^b_{\nu\rho}(-p) \rangle \\
&= i\delta^{ab} \left[\delta_{\mu\nu} p_\rho - \delta_{\mu\rho} p_\nu \right] \quad .
\end{aligned} \tag{8.8}
$$

Feynman rules for the remaining elements of (8.4), required for mass operations occurring inside loops, are derived in a similar fashion.

8.2 Preliminary treatment of Feynman diagrams

Having outlined the general structure of the calculation, the best way of introducing the real nature of the task is by considering the number of Feynman diagrams generated by a Lagrangian including five fields and two couplings. Any investigation beyond the first loop order requires a treatment using automatic generation of Feynman diagrams, and for this we use the QGRAF package, [43]. QGRAF is an algorithm, derived using graph theory, for automatic Feynman diagram generation in any perturbative field theory. Graph theory consists of basic structures called pseudographs: these consist of a set of nodes (vertices) that are joined by edges (lines). Every pseudograph may be represented by an adjacency matrix with entries a_{ij} denoting the number of edges i joining nodes j. There is a direct equivalence between a pseudograph and the underlying topology of a Feynman diagram. Clearly, a Feynman diagram is a more complex structure than a pseudograph, and so the QGRAF algorithm includes a process called colouring. The colouring process assigns different *colours* to the edges of a pseudograph to distinguish between different fields. A consideration of perturbative quantum field theory involves a colouring process that is more complicated than for ordinary graph theory. In particular, anti-commuting fields must be described using orientated edges to reflect directional *charge* flow, and we must also recognize that external particles are distinguishable so that all external nodes must be labeled. Finally, non trivial symmetries will inevitably lead to the generation of equivalent Feynman diagrams during the labeling and colouring procedures. It is desirable that a single *representative* diagram is selected from a class of equivalent diagrams, and this requires the introduction of a suitable elimination procedure. The QGRAF package implements an elimination procedure by constructing the symmetry group of the underlying pseudograph. Elimination criteria are defined using the properties of this group. Given this, it is straightforward to generate multi loop Feynman diagrams for a given Lagrangian by submitting all incidences of propagating fields, defining them to be commuting or anti commuting, and interaction vertices as input for the algorithm. The only additional input required for QGRAF to generate all the Feynman diagrams for a given Green's function at a specific loop order, is information about the external fields and the loop number. A completely fail safe check that QGRAF has produced a correct diagramatical interpretation of a Green's function, for a given Lagrangian, is not possible. When extracting the three loop anomalous dimension for the gauge invariant mass operator using the Lagrangian (8.2), we note that stringent consistency checks exist. These checks include making reference to previously obtained results from the literature at each stage in the calculation and also obtaining a finite result for the renormalization group functions of the new parameters.

Green's function	One loop	Two loop	Three loop	Total
$A^a_\mu \, A^b_\nu$	5	52	1279	1336
$c^a \, \bar{c}^b$	1	8	152	161
$\psi^{iI} \, \bar{\psi}^{jJ}$	1	8	152	161
$B^a_{\mu\nu} \, \bar{B}^b_{\sigma\rho}$	1	20	464	485
$G^a_{\mu\nu} \, \bar{G}^b_{\sigma\rho}$	1	20	464	485
$A^a_\mu \, \bar{B}^b_{\nu\sigma} \, B^c_{\rho\phi}$	7	166	5827	6000
$A^a_\mu \, \mathcal{O} \, B^b_{\nu\sigma}$	5	131	6917	7053
Total	21	405	15255	15681

Table 8.1. Number of Feynman diagrams for each Green's function.

It is necessary to renormalize seven Green's functions to extract the anomalous dimension for the gauge invariant mass operator, the complete list, together with the number of diagrams associated with each Green's function at a given loop order, is given in Table 8.1. Given the large number of diagrams, as with the generation, evaluation is performed automatically using the MINCER algorithm, [67]. This algorithm computes analytically the one, two and three-loop massless integrals of propagator type, 2-point functions. The calculation done here is at the three loop order, for which MINCER considers separately, fourteen generic topologies, T_i, where

$$\mathrm{T}_i \in \{\mathrm{LA}, \mathrm{BE}, \mathrm{NO}, \mathrm{BU}, \mathrm{FA}, \mathrm{Y1}, \mathrm{Y2}, \mathrm{Y3}, \mathrm{Y4}, \mathrm{Y5}, \mathrm{O1}, \mathrm{O2}, \mathrm{O3}, \mathrm{O4}\} \ . \tag{8.9}$$

Before submitting the three loop Green's functions, described using lists of Feynman diagrams, as input for the MINCER algorithm, we must identify all of the diagrams in Table 8.1 with a topology in the set (8.9). Each of the generic topologies in (8.9) describe an underlying structure, compatible with a large number of specific three loop topologies. A complete list of specific topologies is generated, using QGRAF, by making reference to the formal Lagrangian

$$L_{toy} \;=\; (\partial\phi)^2 \;+\; a\phi^3 \;+\; b\phi^4 \,, \tag{8.10}$$

which contains only those features necessary to generate all of the pseudographs (specific topologies) contained within the generic MINCER set. Using (8.10), three loop master diagram lists are generated for 2 and 3-point functions, leading to 49 and 463 distinct topologies respectively. The 3-point function list is used to describe the vertex $\langle A^a_\mu B^b_{\nu\rho} B^c_{\sigma\tau}\rangle$ and also the operator insertion defined using the modified 2-point function $\langle A^a_\mu \mathcal{O} B^b_{\nu\sigma}\rangle$, where the mass insertion is identified using an external node. To make the 3-point function list compatible for use with the MINCER algorithm, it is necessary to nullify the momentum of one external leg. In certain cases, this has the effect of leaving a snail graph embedded inside a *modified 2-point function*, snail graphs give a zero contribution in any massless theory, effectively reducing these graphs to a diagram with a lower loop order. All lower order diagrams were considered as part of the previous treatments, [58],[59], and so 53 such examples were identified and labeled as null diagrams that the algorithm was instructed to disregard.

The process of identifying all remaining diagrams in the master 2 and 3-point lists with a generic topology is best described by making reference to a real master diagram and one of the MINCER topologies. For aesthetic reasons we choose the Benz topology, Fig 8.1, and for simplicity we choose the 3-point master diagram 114, shown graphically

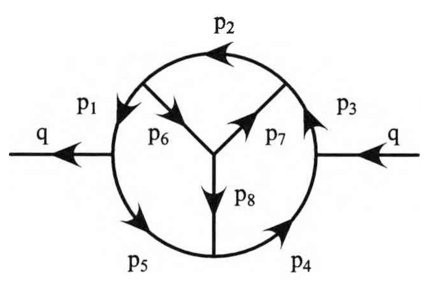


Figure 8.1: MINCER Benz topology

in Fig 8.2, and described below using an exert from the QGRAF output list. There is a large amount of freedom concerning what style is adopted for the output in a QGRAF list file. For the present task, it is sufficient to describe diagrams completely in terms of interaction vertices. Only information about the propagator index is given, where two fields with a common argument belong to the same propagator.

```
list = form ;
lagfile - 'toy' ;
in = phi,phi,phi ;
```

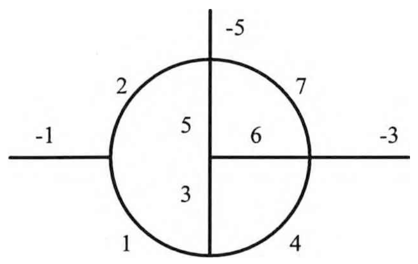


Figure 8.2: 3-point master diagram 114

```
 out = ;
 nloop = 3 ;
 options = nosnail,notadp,onepi ;
#[ d1:
    .
    .
    .
#[ d114:
  1
 *vx(phi(-1),phi(1),phi(2))
 *vx(phi(1),phi(4),phi(3))
 *vx(phi(5),phi(6),phi(3))
 *vx(phi(-5),phi(2),phi(7),phi(5))
 *vx(phi(-3),phi(7),phi(4),phi(6))
#] d114:
    .
    .
    .
#] d463:
end
```

The markers # define the "proper fold" (start and end) of each diagram. All internal (loop) momenta in Fig 8.1 are labeled p_i and the momentum through the 2-point function is labeled by q. All edges, internal lines, in the pseudograph of Fig 8.2 are labeled using positive integers, the external lines are labeled using negative integers, odd numbers define ingoing momentum and even numbers define outgoing momentum, [43].

A single glance tells us that by shrinking to a point the line labeled with momentum p_3 in Fig 8.1, setting $p_3 = 0$, and removing the external line with label -5 in Fig 8.2 gives the two diagrams an identical structure. We label the master 3-point diagram 114 as one specific incidence of a BE type topology and a similar analysis is carried out for all of the other master diagrams. We take this opportunity to emphasize that diagram 114 was chosen for reasons of simplicity, the identification process illustrated using Figs 8.1 and 8.2 is rarely quite so transparent.

Identifying internal edges of pseudographs denoted by positive integers, with the corresponding MINCER loop momenta p_i, is carried out individually for all of the master topologies using converter files. Converter files are incorporated into a set of algorithms, written using the symbolic manipulation language FORM, [44]. The structure of the algorithm is discussed in the next section. The content of the converter files (momentum mapping) will also dictate the Lorentz and colour structure for the completed diagrams. To complete the description of converting a specific topology into a notation suitable for a treatment by the MINCER algorithm, we include an example file that will reconcile the master diagram 114 with the Benz topology.

```
*example file d144 -> be
id vx(phi(-1),phi(1),phi(2))
= vx(phi(-1),phi(1,-p5),phi(2,p1));
id vx(phi(1),phi(4),phi(3))
= vx(phi(1,p5),phi(4,-p4),phi(3,p8));
id vx(phi(5),phi(6),phi(3))
= vx(phi(5,p6),phi(6,-p7),phi(3,-p8));
id vx(phi(-5),phi(2),phi(7),phi(5))
= vx(phi(-5),phi(2,-p1),phi(7,p2),phi(5,-p6));
id vx(phi(-3),phi(7),phi(4),phi(6))
= vx(phi(-3),phi(7,-p2),phi(4,p4),phi(6,p7));
.sort
```

Using a slightly more sophisticated approach, that does not shed any further light on the conversion process and so we choose not to give details here, it is possible to program a version of this file suitable for converting all real Feynman diagrams with a specific topology identical to the master topology of diagram 114. As such, it is possible to furnish all of the real Feynman diagrams of Table 8.1 with 460 labels, 459 specific three loop 2 and 3-point function master topologies and a null flag instructing the algorithm to disregard certain diagrams. The list of three loop Feynman diagrams for each Green's function is edited so that all diagrams are given the appropriate label. After labeling, each diagram can be automatically directed to one of 460 converter files.

8.3 The main program

The next step is construction of the main programs used to create the Feynman diagrams for each Green's function, [67]. The QGRAF output is in the from of a list of basic instructions for a diagram. In order to proceed, we must implement those instructions; add the complete detail of the Feynman rules for each vertex. Subsequently, diagrams are evaluated by applying the appropriate sub routine from the MINCER algorithm. At this stage of the calculation it is not practical to manipulate a list of Feynman *diagrams*, we must abandon the list format and each diagram must exist in its own right. Similarly, manipulating 15,255 diagrams individually is not realistic. A solution is made possible using FORM, [44], in fact this is the essence of symbolic manipulation. The easiest way to explain how the main program uses QGRAF lists to create Feynman diagrams is by using an example main program (dograph) file.

```
*main program
l1 #include declare.h
l2 #include mincer.h
l3 #define SHEME "0"
l4 G 'DIA' =
l5 #include gflist.qg # 'DIA'
l6 #include conv'gt''i'.h
l7 #include frlag.h
l8 #include prelimalg.h
l9 #call integral('gt')
l10 print;
l11 .store
l12 save f'DIA'.res;
l13 .end
```

This file is written using FORM.

- line 1: The first task is to include a file in which all functions, variables, vectors and indices are declared.
- line 2: Diagrams are evaluated using the MINCER algorithm, its contents must also be included.
- line 3: Instructs MINCER that divergences should be removed using the $\overline{\text{MS}}$ scheme.

- line 4: Defines a "Global" expression DIA, global expression can be stored and saved for use in a later program. 'DIA' is left arbitrary using single quotation marks.

- line 5: Details of the global expression DIA are found in the appropriate QGRAF list file which is also included in such a way that only the proper fold is considered.

- line 6: Directs the diagram instructions towards the converter file appropriate to its specific topology (st=gt/i).

- line 7: The diagram is created using the explicit Feynman rules for the Lagrangian (8.2) and furnished with the appropriate Lorentz and colour indices.

- line 8: It is convenient, and may save a lot of computer time later, to do a preliminary treatment of group theory algebra factors at this stage.

- line 9: FORM is instructed to evaluate the diagram using the appropriate MINCER subroutine (gt).

- line 10: When the diagram reaches its intermediate point, we may want to look at it.

Lines 11, 12 and 13, store the diagram, save it, in for example the file fd114.res, and terminate the program. Leaving the identity of the global expression in the main program open allows us to use the make utility, a long time mainstay of the UNIX tool set. Turning a list of diagram instructions into a complete set of evaluated diagrams stored using individual files requires a second list, details of which are included in a makefile, [67]. An example makefile might look like this.

```
♯makefile
DIAS = fdia1.res fdia2.res fdia3.res fdia4.res fdia5.res \
       fdia6.res fdia7.res fdia8.res fdia9.res fdia10.res \
sumgraph.sav:  $(DIAS) sumgraph
 form -l sumgraph
fdia1.res:  dograph
 form -d DIA=dia1 dograph > dia1.log
    .
    .
    .
fdia10.res:  dograph
 form -d DIA=dia10 dograph > dia10.log
```

The makefile actually includes two lists. A list of files, `DIAS`, and a list of simple instructions on how to construct them, refer to the main program, dograph, replacing `DIA` with the desired diagram name/number. The only thing to be done now is to give the make command so that all files in the list `DIAS` are created by accessing the main program. The appearance of `sumgraph` in the makefile is self explanatory, we require a third program to recombine the results for the evaluated diagrams back in to a Green's function. The `sumgraph` file is where the final group theory algebra modules are implemented before doing the renormalization, also in this file.

For completeness, it is worthwhile to conclude this section by considering, briefly, how the MINCER algorithm, [67], evaluates 2-point functions composed of massless propagators. This is explained most clearly by making explicit reference to one and two loop integrals only. The basic p integral, where p describes the loop momentum that is to be integrated over, is shown in Fig. 8.3. The integrand consists of powers, p^2 and $(q-p)^2$, in the denominator, and an an arbitrary number of individual momentum, p_{μ_i}, in the numerator. Integration is done in d-dimensional space, where $d = 4 - 2\epsilon$. In addition, it is assumed that powers of p^2 and $(q-p)^2$ are non-integer and may contain ϵ. In particular, this will be the case when two loop integrals are reduced to convolutions

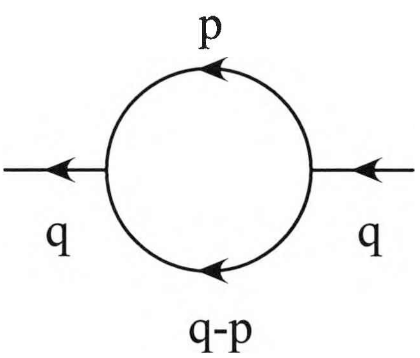


Figure 8.3: The basic one loop diagram

of one loop integrals. The **general** solution for such one loop integrals is known and may be summarized using a single formula, [68],

$$\int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\mathcal{P}_n(p)}{p^{2\alpha}(q-p)^{2\beta}} =$$

$$\frac{1}{(4\pi)^2}(q^2)^{d/2-\alpha-\beta} \sum_{\sigma \geq 0}^{[n/2]} G(\alpha,\beta,n,\sigma) q^{2\sigma} \left\{ \frac{1}{\sigma!} \left(\frac{\Box}{4} \right)^{\sigma} \mathcal{P}_n(p) \right\} \Big|_{p=q} , \quad (8.11)$$

where $\Box = \partial p_\mu \partial p_\mu$ and G is given in terms of Γ-functions

$$G(\alpha,\beta,n,\sigma) = (4\pi)^{\epsilon} \frac{\Gamma(\alpha+\beta-\sigma-d)\Gamma(d/2-\alpha+n-\sigma)\Gamma(d/2-\beta+\sigma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(d-\alpha-\beta+n)} . \quad (8.12)$$

A detailed consideration of the MINCER algorithm is included in [68], we simply note here that the general solution to a one loop integral is expressed using Γ-functions. The Γ-function may be expressed using expansion formulae, for example

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\epsilon} + \psi_1(n+1) + O(\epsilon)\right] \ , \tag{8.13}$$

where

$$\psi_1(n+1) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \gamma_E \ , \tag{8.14}$$

and γ_E is the Euler-Mascheroni constant. It also obeys a number of recursion relations, such as

$$\Gamma(-1+\epsilon) = \frac{-1}{1-\epsilon}\Gamma(\epsilon) \ . \tag{8.15}$$

Given this, the authors of MINCER, [67], are able to exploit powerful programming features included in FORM, [44], to produce a reliable and efficient algorithm which is able to compute, analytically, a wide range of one loop integrals with general structure (8.11). The requirement that we should be able to evaluate, automatically, such a broad scope of one loop integrals, stems from the fact that these are, in general, generated by convolutions of two and three loop integrals. At two loops there are three distinct topologies, T1, T2 and T3. Starting with topology T2, Fig 8.4, we note that this is in fact a convolution of two one loop integrals, when the inner integral is evaluated the outer line has a power including ϵ. The T3 topology, Fig 8.5, is easier still, it is simply

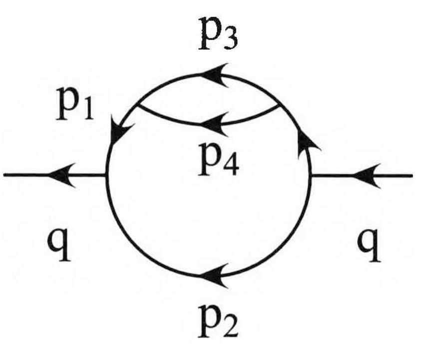


Figure 8.4: T2 topology

a direct product of two one loop integrals. Unlike T2, the topology T1, Fig 8.6, has no trivial sub-graph, and at first sight appears to be an irreducible integral that will have to be evaluated using a different method. In fact, there does exist a rather simple method, [68],[69], for reducing integrals with topology T1 to simpler integrals of type T2 and T3. This method relies on the existence of identities, obtained using integration by parts within dimensional regularization, and is summarized using the triangle rule, Fig 8.7, a non trivial sub-graph of the integral T1. Using the triangle rule, we observe

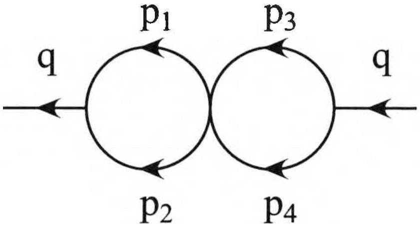


Figure 8.5: T3 topology

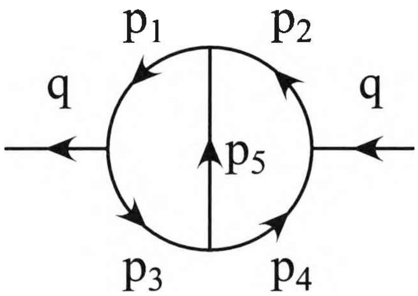


Figure 8.6: T1 topology

that a sub integral for the integral with topology T1, is given by

$$\mathrm{I}(\alpha_0,\beta_1,\alpha_1,\beta_2,\alpha_2)=\int_p\frac{1}{(p^2)^{\alpha_0}[(p+k)^2]^{\beta_1}(k^2)^{\alpha_1}[(p+k+q)^2]^{\beta_2}[(k+q)^2]^{\alpha_2}}\ . \tag{8.16}$$

It is possible to study (8.16) using the integration by parts identity

$$\int_p\frac{\partial}{\partial p_\mu}\left\{\frac{p_\mu}{(p^2)^{\alpha_0}[(p+k)^2]^{\beta_1}(k^2)^{\alpha_1}[(p+k+q)^2]^{\beta_2}[(k+q)^2]^{\alpha_2}}\right\}=0\ , \tag{8.17}$$

where a little algebra produces the recursion relation

$$\begin{aligned}\mathrm{I}(\alpha_0,\beta_1,\alpha_1,\beta_2,\alpha_2) \;=\;& [\beta_1(\mathrm{I}(\alpha_0-1,\beta_1+1,\alpha_1,\beta_2,\alpha_2)-\mathrm{I}(\alpha_0,\beta_1+1,\alpha_1-1,\beta_2,\alpha_2))\\ &\beta_2(\mathrm{I}(\alpha_0-1,\beta_1,\alpha_1,\beta_2+1,\alpha_2)-\mathrm{I}(\alpha_0,\beta_1,\alpha_1,\beta_2+1,\alpha_2-1))]\\ &/(d-2a_0-\beta_1-\beta_2)\ .\end{aligned} \tag{8.18}$$

This recursion relation can be used to remove integer powers of α_i, when α_0 is eliminated we obtain an integral of type T3, and when α_1 or α_2 is eliminated we get an integral of type T2, both convolutions of one loop integrals suitable for evaluation using the algorithm. This recursion relation is the cornerstone for a complete treatment of two and three loop integrals. Finally, at the end of the calculation, results are converted into the standard $\overline{\mathrm{MS}}$ scheme.

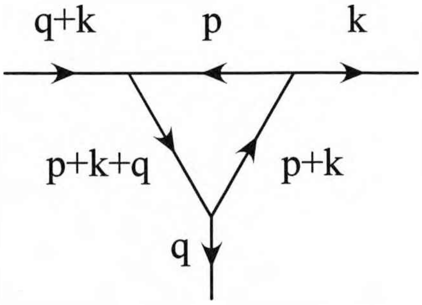


Figure 8.7: The triangle sub-graph

8.4 Group Theory Algebra

Integrating all of the three loop Feynman diagrams describing each Green's function automatically using MINCER effectively reduces the remaining task to a consideration of group theory factors. The purpose of our calculation is to extract the three loop $\overline{\text{MS}}$ renormalization constants for the Lagrangian (8.2). Given this, from here onwards we consider only the divergent parts of each Green's function. A number of packages designed to tackle colour group theory factors typical of high loop order calculations are available, [70]. Those packages are specifically geared towards treating group theory factors commonly occurring in a wide range of calculations described by a QCD/Yang-Mills type Lagrangian not containing a quartic tensor coupling λ^{abcd} with explicit colour structure. As such, we choose not to incorporate them into this calculation and treat divergent parts of all Green's functions using a more restricted set of algorithms. Some of the algorithms were developed prior to this work and have been tested against existing results in earlier calculations. We also develop new treatments specifically designed to tackle factors commonly occurring in different parts of this calculation.

In the absence of a quartic tensor coupling, individual diagrams associated with any QCD type calculation at high loop order have a complicated group structure, given in terms of the structure constants f^{abc}, coming from the gauge-ghost and gauge self couplings, and group generators T^a_R, from the gauge-quark coupling. The objects f^{abc} and T^a_R, are respectively, the group structure constants and, group generators (for the representation R), for a semi-simple classical Lie group. In the absence of a tensor coupling, the group structure of an individual three loop diagram is given by products of these tensor objects including a large number of internal indices. The object of any algorithm designed to tackle group theory factors associated with QCD calculations at high loop order is to reduce the total sum of divergent contributions to a Green's function into a single expression with the same simple group structure as the tree level object, this property follows directly from renormalizability. The final expression is given in terms of group invariants, [74]. This is achieved through cancellations between tensor products from different diagrams and tensor contractions.

The group generators satisfy a commutation relation given by

$$\left[\tau^a, \tau^b\right] = i f^{abc} \tau^c , \tag{8.19}$$

and generators of the adjoint representation are related to the structure constants through the expression

$$(\tau^a_A)_{bc} = -i f^{abc} . \tag{8.20}$$

The commutation relation (8.19) obeys a Jacobi identity which may be expressed completely in terms of the group structure constants and is given by,

$$f^{man} f^{mbc} + f^{mbn} f^{amc} + f^{mcn} f^{abm} = 0 . \tag{8.21}$$

Tensor contractions are achieved using the quadratic Casimir operator C_R defined by

$$(\tau^a_R \tau^a_R)_{ij} = C_R \delta_{ij} , \tag{8.22}$$

where in the adjoint representation (8.20) we obtain the tensor contraction

$$f^{acd} f^{bcd} = C_A \delta^{ab} . \tag{8.23}$$

Using the Jacobi identity (8.21) and the contraction (8.23) leads to the expression

$$f^{apq} f^{bpr} f^{cqr} = \frac{1}{2} C_A f^{abc} . \tag{8.24}$$

Making use of the above relations, products of group generators are contracted using expressions with the form

$$\begin{aligned} \tau^b_R \tau^a_R \tau^a_R \tau^c_R &= C_R \tau^b_R \tau^c_R \\ \tau^a_R \tau^b_R \tau^a_R &= \left(C_R - \frac{1}{2} C_A \right) \tau^b_R . \end{aligned} \tag{8.25}$$

We also make use of the identity

$$\mathrm{tr}(\tau^a_R \tau^b_R) = \delta^{ab} T_R . \tag{8.26}$$

Not used as part of this calculation but included here for completeness, we note that it is sometimes necessary to appeal to a symmetrized trace of group generators where, for four generators in the adjoint representation the totally symmetric trace is defined using the identity,

$$d^{abcd}_A = \frac{1}{6} \mathrm{tr} \left(T^a_A T^{(b}_A T^c_A T^{d)}_A \right) . \tag{8.27}$$

Finally, it is worthwhile to recall the antisymmetric property of the group structure constants such that

$$f^{abb} = 0 . \tag{8.28}$$

When a quartic tensor coupling is not present, a suitable combination of the identities and contractions described above, incorporated into a group theory algorithm, is sufficient to reduce the total sum of divergent loop corrections to a Green's function, derived using a renormalizable Lagrangian, to a single expression with the same simple group structure as the tree level object.

The coupling λ^{abcd} is described by an invariant tensor of rank four in the adjoint representation with symmetry constraints,

$$\lambda^{abcd} = \lambda^{cdab} \tag{8.29}$$

$$\lambda^{abcd} = \lambda^{bacd} \,. \tag{8.30}$$

Invariance is expressed through the identity

$$\lambda^{abcd} = U_A^{ap} U_A^{bq} U_A^{cr} U_A^{ds} \lambda^{pqrs} \quad , \tag{8.31}$$

where U_A^{ap} is an element of the group in the adjoint representation such that, in the infinitesimal case, $U_A^{ap} = \exp\left[i\omega^n (\tau_A)^n_{ap}\right]$. Hence, infinitesimally

$$\begin{aligned}
\lambda^{abcd} &= (\delta^{ap} + \omega^n f^{nap})(\delta^{bq} + \omega^n f^{nbq})(\delta^{cr} + \omega^n f^{ncr})(\delta^{ds} + \omega^n f^{nds})\lambda^{pqrs} + O(\omega^2) \\
&= \Big(\delta^{ap}\delta^{bq}\delta^{cr}\delta^{ds} + \delta^{bq}\delta^{cr}\delta^{ds}\omega^n f^{nap} + \delta^{ap}\delta^{cr}\delta^{ds}\omega^n f^{nbq} \\
&\qquad + \delta^{ap}\delta^{bq}\delta^{ds}\omega^n f^{ncr} + \delta^{ap}\delta^{bq}\delta^{cr}\omega^n f^{nds}\Big)\lambda^{pqrs} + O(\omega^2) \\
&= \lambda^{abcd} + \omega^n f^{nap}\lambda^{pbcd} + \omega^n f^{nbq}\lambda^{aqcd} + \omega^n f^{ncr}\lambda^{abrd} + \omega^n f^{nds}\lambda^{abcs} + O(\omega^2) \,.
\end{aligned} \tag{8.32}$$

Re-labeling all dummy indices, $\{p, q, r, s\} \to m$, gives a generalized Jacobi identity,

$$f^{man}\lambda^{mbcd} + f^{mbn}\lambda^{amcd} + f^{mcn}\lambda^{abmd} + f^{mdn}\lambda^{abcm} = 0 \,, \tag{8.33}$$

for the quartic tensor coupling. We also make use of the identities,

$$\lambda^{acde}\lambda^{bcde} = \frac{1}{N_A}\delta^{ab}\lambda^{cdpq}\lambda^{cdpq} \,, \quad \lambda^{acde}\lambda^{bdce} = \frac{1}{N_A}\delta^{ab}\lambda^{cdpq}\lambda^{cpdq} \tag{8.34}$$

which follow from the fact that there is only one rank two invariant tensor in a classical Lie group. Analogous identities for the extension to the product of three tensor couplings, for example

$$\lambda^{acde}\lambda^{bpdq}\lambda^{cqep} = \frac{1}{N_A}\delta^{ab}\lambda^{rcde}\lambda^{rpdq}\lambda^{cqep} \quad , \tag{8.35}$$

have also been used.

A suitable combination of all the identities and contractions described above, including those for the quartic tensor coupling, incorporated into a new group theory algorithm, is sufficient to reduce the total sum of divergent loop corrections to the Green's functions, derived using the Lagrangian (8.2), to a single expression with the same simple group structure as the tree level object.

Before applying symmetry operations, it is sensible to begin a new group theory algorithm by addressing the inevitable random spread of indices that result from a large number of diagrams. Up to two loop order, this was done by making use of the FORM *wild-carding* technique. Wild-cards are generic objects that are used in patterns and can then match a class of objects. For example,

```
id ff(A1?,A2?,A4?)*lambda(A3?,A4?,A5?,A2?) =
  -ff(A1,A2,A4)*lambda(A3,A2,A5,A4); .
```

All the indices in this tensor product are described using wild-cards, as such the symmetry operation described is applied to all structures with this pattern. Pattern matching is effective for mapping a class of group structures described using a limited number of indices onto a single structure. The expressions considered as part of the three loop calculation contain up to nine pairs of indices and are described by a larger number of different, more complicated, group structures. A summary of the different tensor object combinations and the number of index pairs required to describe the individual group structures for the three loop corrections to the $\langle B^a_{\mu\nu}\bar{B}^b_{\rho\sigma}\rangle$ and $\langle A^a_\mu B^b_{\nu\rho}\bar{B}^c_{\sigma\tau}\rangle$ n-point functions is given in Table 8.2, where the structure constants and the quartic tensor coupling are simply denoted by f and λ respectively. Much early effort was

Green's function	Three loop tensor objects	Index pairs
$B^a_{\mu\nu}\bar{B}^b_{\sigma\rho}$	$\lambda\lambda\lambda$	5
	$ff\lambda\lambda$	6
	$ffff\lambda$	7
	$ffffff$	8
$A^a_\mu\bar{B}^b_{\nu\sigma}B^c_{\rho\phi}$	$f\lambda\lambda\lambda$	6
	$fff\lambda\lambda$	7
	$fffff\lambda$	8
	$fffffff$	9

Table 8.2 Three loop tensor combinations and number of index pairs

spent trying to extend the wild-carding techniques of the two loop calculation to three loop order, with a limited degree of success. Applying wild-carding techniques to small sections of larger group structures did map a large number of random index patterns onto a relatively small number of common forms. However, mapping this small spread of common forms onto a group structure described by a single index pattern required additional, extensive, re-labeling processes. When considering the large number of different group structures possible for the combinations of tensor objects described in

Table 8.2, then one can begin to imagine how inefficient and potentially error prone an initial treatment using FORM wild-carding techniques was. An additional complicating factor was the fact that certain parts of the three loop calculation, group structures described by up to six index pairs, were amenable to the wild-carding techniques of the two loop calculation. After several unsuccessful attempts to persevere with a naive extension of the techniques used in the two loop calculation to group structures described by more than six index pairs, it was decided to postpone consideration of such group structures until after a wholesale re-labeling process had been carried out. This re-labeling process simply involved re-ordering index pairs into, as close as possible, ascending order around the external indices, within the symmetry constraints dictated by the different combinations of the structure constant f^{abc}, and tensor coupling λ^{abcd}. After this process was complete, it was possible to use appropriate combinations of the symmetry properties described above to reduce a large spread of individual three loop group structures describing the Green's functions to a set of objects given by two loop structures.

Having reduced all of the three loop group structures to a smaller set described in terms of two loop structures, one might have thought that it would be possible to simply re-introduce the code used to evaluate the two loop calculation and arrive at a result. Whilst the remaining two loop group structures are indeed less complicated, application of the wild-carding techniques remained somewhat problematic because of the random way in which the contractions (8.23) and (8.24) remove index pairs from the three loop structures. Resulting individual two loop structures did not contain more than six index pairs. However, common two loop group structures described by different random spreads of paired indices were still described, collectively, by more than six pairs. This limited the effectiveness of the FORM wild-carding technique and was also a possible source of confusion. Once again, it was decided to carry out a wholesale re-labeling process before a consideration of the newly generated two loop group structures. This time, the re-labeling process meant ensuring that all common two loop group structures were described by the same index pairs. The tensor combinations and number of index pairs are given in Table 8.3. In order to get all of the remaining two loop tensor objects onto the appropriate set of paired indices requires an extensive marking process. This is done by attributing flags to each different index pair, $h_1, .., h_9$, and an additional flag, $g_3, .., g_6$, to identify how many pairs are necessary to fully label a specific tensor product at two loops, Table 8.3. For example,

```
if ( count(ff,1) = 3) && ( count(lambda,1) = 1);
multiply g5;
endif;
```

Green's function	Two loop tensor objects	Index pairs
$B^a_{\mu\nu}\,\bar{B}^b_{\sigma\rho}$	$\lambda\lambda$	3
	$ff\lambda$	4
	$ffff$	5
$A^a_\mu\,\bar{B}^b_{\nu\sigma}\,B^c_{\rho\phi}$	$f\lambda\lambda$	4
	$fff\lambda$	5
	$fffff$	6

Table 8.3. Two loop tensor combinations and number of index pairs

```
id ff(?a,A1,?b)*ff(?c,A1,?d) = h1*ff(?a,A1,?b)*ff(?c,A1,?d);
id ff(?a,A1,?b)*lambda(?c,A1,?d) = h1*ff(?a,A1,?b)*lambda(?c,A1,?d);
id lambda(?a,A1,?b)*lambda(?c,A1,?d) = h1*lambda(?a,A1,?b)*lambda(?c,A1,?d);
      .
      .
      .
id ff(?a,A9,?b)*ff(?c,A9,?d) = h9*ff(?a,A9,?b)*ff(?c,A9,?d);
id ff(?a,A9,?b)*lambda(?c,A9,?d) = h9*ff(?a,A9,?b)*lambda(?c,A9,?d);
id lambda(?a,A9,?b)*lambda(?c,A9,?d) = h9*lambda(?a,A9,?b)*lambda(?c,A9,?d);
.sort
```

where we have used the symbol `(?a,A1,?b)` to define an argument *field* that may contain an arbitrary number of arguments. After marking

$$f^{a56} f^{b57} f^{c34} \lambda^{3746} = g_5 h_3 h_4 h_6 h_7 h_8 f^{a68} f^{b78} f^{c34} \lambda^{3746} , \tag{8.36}$$

and where necessary, index pairs are systematically re-labeled with a smaller index number by making reference to the flags, for example

```
if ( count(g5,1) = 1 );
if ( count(h8,1) = 1 && count(h1,1) = 0 );
id ff(?a,A8,?b)*lambda(?c,A8,?d) = h1/h8*ff(?a,A1,?b)*lambda(?c,A1,?d);
id ff(?a,A8,?b)*ff(?c,A8,?d) = h1/h8*ff(?a,A1,?b)*ff(?c,A1,?d);
endif;
if ( count(h7,1) = 1 && count(h1,1) = 0 );
```

```
id ff(?a,A7,?b)*lambda(?c,A7,?d) = h1/h7*ff(?a,A1,?b)*lambda(?c,A1,?d);
id ff(?a,A7,?b)*ff(?c,A7,?d) = h1/h7*ff(?a,A1,?b)*ff(?c,A1,?d);
endif;
if ( count(h6,1) = 1 && count(h1,1) = 0 );
id ff(?a,A6,?b)*lambda(?c,A6,?d) = h1/h6*ff(?a,A1,?b)*lambda(?c,A1,?d);
id ff(?a,A6,?b)*ff(?c,A6,?d) = h1/h6*ff(?a,A1,?b)*ff(?c,A1,?d);
endif;
                    .
                    .
                    .
if ( count(h7,1) = 1 && count(h5,1) = 0 );
id ff(?a,A7,?b)*lambda(?c,A7,?d) = h5/h7*ff(?a,A5,?b)*lambda(?c,A5,?d);
id ff(?a,A7,?b)*ff(?c,A7,?d) = h5/h7*ff(?a,A5,?b)*ff(?c,A5,?d);
endif;
if ( count(h6,1) = 1 && count(h5,1) = 0 );
id ff(?a,A6,?b)*lambda(?c,A6,?d) = h5/h6*ff(?a,A5,?b)*lambda(?c,A5,?d);
id ff(?a,A6,?b)*ff(?c,A6,?d) = h5/h6*ff(?a,A5,?b)*ff(?c,A5,?d);
endif;
endif;                                                          (8.37)
```

such that

$$g_5 h_3 h_4 h_6 h_7 h_8 f^{a68} f^{b78} f^{c34} \lambda^{3746} = g_5 h_1 h_3 h_4 h_6 h_7 f^{a61} f^{b71} f^{c34} \lambda^{3746} \tag{8.38}$$

$$g_5 h_1 h_3 h_4 h_6 h_7 f^{a61} f^{b71} f^{c34} \lambda^{3746} = g_5 h_1 h_2 h_3 h_4 h_6 f^{a61} f^{b21} f^{c34} \lambda^{3246} \tag{8.39}$$

$$g_5 h_1 h_2 h_3 h_4 h_6 f^{a61} f^{b21} f^{c34} \lambda^{3246} = g_5 h_1 h_2 h_3 h_4 h_5 f^{a51} f^{b21} f^{c34} \lambda^{3245} \; . \tag{8.40}$$

A similar process is carried out for tensor products described by 3,4 and 6 paired indices until all possible groups of two loop tensor objects are given in terms of the appropriate minimal set of index pairs. Notice that introducing the flag h_1 in the substitution (8.37) means that FORM does not apply, incorrectly, the further possible substitutions, {A7,A6} $\to$ A1 to the tensor object (8.38). The introduction of so many flags to label groups of tensor objects does look elaborate, on the page and on the computer screen. However, implementing this re-labeling process efficiently requires use of the FORM *count* statement. The effectiveness of the symbolic manipulation language FORM, or indeed any other programming language, derives from its simplicity. As such, the count statement does not search inside function arguments, in this case tensor objects, and so all of the flags are necessary. Although it may seem like overkill to take so much care over an apparently trivial labeling task, it is worth stressing that an effective application of the Jacobi identities, (8.21) and (8.33), necessary to achieve the further cancellations and subsequent contractions required to reduce all remaining factors to one loop structures, is severely hampered by an unnecessarily large spread of index pairs. After the re-labeling task is complete, all of the flags are set equal to 1 and the rest of the calculation proceeds in a text book manner.

8.5 Renormalization

In order to extract the divergences associated with the loop corrections to a given n-point function, it is necessary to give careful consideration to any correlation function with a specific Lorentz tensor structure. In an arbitrary linear covariant gauge, the gluon propagator is given by

$$\langle A^a_\mu(p) A^b_\nu(-p)\rangle \; = \; -\frac{\delta^{ab}}{p^2}\left[\delta_{\mu\nu} - (1-\alpha)\frac{p_\mu p_\nu}{p^2}\right] \; , \tag{8.41}$$

where the transverse and longitudinal parts must be projected out using a matrix $\mathcal{M}_{TL}$ and considered separately. Before extracting the divergences specific to each component the matrix $\mathcal{M}_{TL}$ is derived by first constructing the matrix

$$\mathcal{N}_{TL} = \begin{pmatrix} \mathcal{P}^{(T)}_{\{\mu|\nu\}}(p)\mathcal{P}^{(T)}_{\{\mu|\nu\}}(p) & \mathcal{P}^{(T)}_{\{\mu|\nu\}}(p)\mathcal{P}^{(L)}_{\{\mu|\nu\}}(p) \\ \mathcal{P}^{(L)}_{\{\mu|\nu\}}(p)\mathcal{P}^{(T)}_{\{\mu|\nu\}}(p) & \mathcal{P}^{(L)}_{\{\mu|\nu\}}(p)\mathcal{P}^{(L)}_{\{\mu|\nu\}}(p) \end{pmatrix} \tag{8.42}$$

where the transverse projection is given by

$$\mathcal{P}^{(T)}_{\{\mu|\nu\}}(p) = \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \; , \tag{8.43}$$

and for the longitudinal projection

$$\mathcal{P}^{(L)}_{\{\mu|\nu\}} = \frac{p_\mu p_\nu}{p^2} \; . \tag{8.44}$$

The matrix $\mathcal{M}_{TL}$ is given by the inverse of $\mathcal{N}_{TL}$, whose elements are polynomials of the dimension d due to contraction of Lorentz indices. Having derived the appropriate form of the matrix $\mathcal{M}_{TL}$, it is used to project out, for example, the transverse piece by multiplying the gluon correlation function with the projection

$$\mathcal{M}_{TL}\mathcal{P}^{(T)}_{\{\mu|\nu\}}(p) \; , \tag{8.45}$$

The explicit form of the matrix is given by

$$\mathcal{M}_{TL} = \frac{1}{(d-1)}\begin{pmatrix} 1 & 0 \\ 0 & (d-1) \end{pmatrix} \; . \tag{8.46}$$

For the tensor fields with propagator type

$$\langle B^a_{\mu\nu}(p)\bar{B}^b_{\sigma\rho}(-p)\rangle \;=\; -\frac{\delta^{ab}}{2p^2}\left[\delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\sigma}\right] \;, \tag{8.47}$$

we adopt a similar method. The most general tensor propagator is separated into components 1 and 2 according to the prescription,

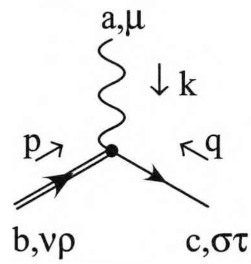
$$\begin{aligned}
\mathcal{P}^{(1)}_{\{\mu\nu|\rho\sigma\}}(p) &= \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} \\
\mathcal{P}^{(2)}_{\{\mu\nu|\rho\sigma\}}(p) &= \delta_{\mu\rho}\frac{p_\nu p_\sigma}{p^2} - \delta_{\mu\sigma}\frac{p_\nu p_\rho}{p^2} - \delta_{\nu\rho}\frac{p_\mu p_\sigma}{p^2} + \delta_{\nu\sigma}\frac{p_\mu p_\rho}{p^2} \;,
\end{aligned} \tag{8.48}$$

where antisymmetry causes terms with the general form $p_\mu p_\nu p_\rho p_\sigma$ to cancel. This approach, where the components, $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$, are now not orthogonal, leads to the analogous matrix

$$\mathcal{M}_{12} = \frac{1}{4(d-1)(d-2)}\begin{pmatrix} 2 & -2 \\ -2 & d \end{pmatrix} \;, \tag{8.49}$$

for the tensor fields which is used to project out component 1 from the 2-point correlation functions.

The remaining Lorentz structures required for this calculation are given by the mass operator insertion, (8.8), and the Feynman rule for the vertex, $\langle A^a_\mu \bar{B}^b_{\nu\rho} B^c_{\sigma\tau}\rangle$, detailed below



$$\begin{aligned}
&= \langle A^a_\mu \bar{B}^b_{\nu\rho} B^c_{\sigma\tau}\rangle \\
&= igf^{abc}\left[\delta_{\nu\sigma}\delta_{\rho\tau}\left(\frac{1}{2}k - q\right)_\mu \right.\\
&\qquad\left. - \delta_{\nu\tau}\delta_{\rho\sigma}\left(\frac{1}{2}k - q\right)_\mu\right] \;.
\end{aligned} \tag{8.50}$$

For the remaining cases, it is not necessary to consider the most general decomposition by deriving a complete matrix. The appropriate, scalar, renormalization factor is obtained by multiplying the correlation function by the same Lorentz structure and inverse square. That is, for the mass insertion, we multiply the correlation function by a factor

$$\frac{[\delta_{\mu\nu}p_\rho - \delta_{\mu\rho}p_\nu]}{(\delta_{\sigma\tau}p_\pi - \delta_{\sigma\pi}p_\tau)^2} \;=\; \frac{[\delta_{\mu\nu}p_\rho - \delta_{\mu\rho}p_\nu]}{2p^2(d-1)} \;. \tag{8.51}$$

The explicit form of the divergences are removed following the procedure of [71]. Using this method, Feynman integrals describing radiative corrections to a correlation/Green's function are evaluated using dimensional regularization and added together in the `sumgraph` program. Independent loop divergences are removed by requiring that poles in ϵ cancel using a combined, multiplicative, renormalization constant,

$$\Gamma = Z_\Gamma \Gamma_o \ , \tag{8.52}$$

where,

$$Z_\Gamma = \prod_i Z_i \ , \tag{8.53}$$

and

$$Z_i = 1 + \sum_i \frac{a_i}{\epsilon^i} \ . \tag{8.54}$$

Multiplicative renormalization constants, Z_i, describe divergent quantities and include factors $1/\epsilon$ and $1/\epsilon^2$, which, when multiplied with finite and infinitesimal contributions to a correlation Γ function describe divergences resulting from sub-graphs at the next order in perturbation theory. In fact, they coincide exactly with all divergent contributions stemming from sub-graphs at the next order in perturbation theory. By alternately adding and subtracting radiative corrections for a Green's function at each different order in perturbation theory, and retaining integral expansions for correlation functions to infinitesimal orders, $O(\epsilon^n)$, it is possible to consider independent loop divergences only.

Before proceeding to the results of the three loop calculation, it is necessary to make a few remarks about the renormalization of the quartic tensor coupling λ^{abcd}. This interaction was identified as forming part of the most general counter term Σ_Δ, by the algebraic renormalization procedure. It is included so that the Lagrangian (8.2) is stable against all radiative corrections (renormalizable). For the present task, calculation of the anomalous dimension of the gauge invariant mass operator

$$\mathcal{O} = \left(B^a_{\mu\nu} - \bar{B}^a_{\mu\nu} \right) F^a_{\mu\nu} \tag{8.55}$$

using the $\overline{\text{MS}}$ renormalization scheme, the tensor coupling is only included where necessary. At one loop order the operator anomalous dimension is extracted from the divergent part of a 2-point function that has no contribution from the tensor coupling. None of the Green's functions used in the one loop calculation make any reference to λ^{abcd}, and it may be regarded as being formally absent. At two-loops the 2-point function does include the quartic tensor coupling. However, it is possible to describe the operator anomalous dimension, and all other Green's functions used to derive it, using a bare (tree level) coupling, λ^{abcd}_o. This particular approach is forced upon us by the special technical difficulties presented by any quartic interaction at higher loop orders, even when a tensor structure is absent. Only for the three-loop calculation is it necessary to have explicit knowledge of the one loop structure of the quartic tensor interaction and this problem was solved prior to starting this calculation, [59]. Its one loop renormalization is described using a tensor function

$$\lambda_o^{abcd} = \lambda^{abcd} + z_\lambda^{abcd}\frac{1}{\epsilon} \,, \tag{8.56}$$

where

$$\begin{aligned} z_\lambda^{abcd} &= -12f_4^{abcd}C_A a^2 + 48 f_4^{apbq} f_4^{cpdq} a^2 - 6\lambda^{abcd} C_A a \\ &+ \frac{1}{8}\left(\lambda^{abpq}\lambda^{cqdp} + \lambda^{apbq}\lambda^{cdpq} + \lambda^{apcq}\lambda^{bpdq} + \lambda^{apdq}\lambda^{bpcq}\right) \,. \end{aligned} \tag{8.57}$$

Also, the one loop β-function for the tensor coupling λ^{abcd} was calculated to be, [59],

$$\begin{aligned} \beta_\lambda^{abcd}(a,\lambda) &= \frac{1}{2}(d-4)\lambda^{abcd} - 6C_A\lambda^{abcd}a - 12C_A f_4^{abcd}a^2 + 48C_A f_4^{apbq} f_4^{cpdq} a^2 \\ &+ \frac{1}{8}\left[\lambda^{abpq}\lambda^{cpdq} + \lambda^{apbq}\lambda^{cdpq} + \lambda^{apcq}\lambda^{bpdq} + \lambda^{apdq}\lambda^{bpcq}\right] \\ &+ O(a^3,\lambda^3) \,. \end{aligned} \tag{8.58}$$

8.6 Three loop results

The three loop $\overline{\text{MS}}$ results given here extend the one and two loop calculations presented in [58] and [59]. We begin with a consideration of the three loop corrections to the 2-point functions for the fields of standard Quantum Chromodynamics. That is, the gluon, quark and Faddeev-Popov ghost 2-point functions. The auxiliary fields, $\{B^a_{\mu\nu}, \bar{B}^a_{\mu\nu}, G^a_{\mu\nu}, \bar{G}^a_{\mu\nu}\}$, and also the quartic tensor coupling, λ^{abcd}, in the massless part of the Lagrangian (8.2) are all objects associated with localization of the mass operator and subsequent algebraic renormalization. When the mass operator is absent, $m = 0$, it should be expected that any additional interactions between these objects and those of ordinary Yang-Mills theory play a completely passive role in the renormalization procedure of the original Yang-Mills/QCD objects. In other words, renormalization of the gluon, Faddeev-Popov ghost and quark fields, subject to the complete set of interactions in the massless Lagrangian, should be identical to those obtained using the ordinary QCD Lagrangian. Using the method of [71] described earlier, the explicit form of the $\overline{\text{MS}}$ renormalization constants required to cancel the poles in ϵ from the three loop corrections to each 2-point are given by

$$\begin{aligned} Z_A &= 1 + \frac{a}{\epsilon}\left[\left(\frac{13}{6} - \frac{\alpha}{2}\right)C_A - \frac{4}{3}T_F N_f\right] \\ &+ \frac{a^2}{\epsilon}\left[\left(\frac{59}{16} - \frac{11\alpha}{16} - \frac{\alpha^2}{8}\right)C_A^2 - \frac{5}{2}C_A T_F N_f - 2C_F T_F N_f\right] \\ &+ \frac{a^2}{\epsilon^2}\left[\left(-\frac{13}{8} - \frac{17\alpha}{24} + \frac{\alpha^2}{4}\right)C_A^2 + \left(1 + \frac{2\alpha}{3}\right)C_A T_F N_f\right] + O(g^6) \,. \\ &+ \frac{a^3}{\epsilon}\left[\left(\frac{9965}{864} - \frac{3\zeta(3)}{16} - \frac{167\alpha}{96} - \frac{\zeta(3)\alpha}{4} - \frac{11\alpha^2}{32} - \frac{\zeta(3)\alpha^2}{16} - \frac{7\alpha^3}{96}\right)C_A^3 \right. \end{aligned}$$

$$
\begin{aligned}
&+\left(-\frac{911}{54}+\frac{\zeta(3)}{6}+\frac{2\alpha}{3}\right)N_fT_FC_A^2+\left(-\frac{5}{54}-8\zeta(3)\right)N_fT_FC_FC_A \\
&+\frac{2}{3}N_fT_FC_F^2+\frac{76}{27}N_f^2T_F^2C_A+\frac{44}{27}N_f^2T_F^2C_F\Bigg] \\
&+\frac{a^3}{\epsilon^2}\Bigg[\left(-\frac{7957}{864}-\frac{143\alpha}{96}+\frac{13\alpha^2}{24}+\frac{7\alpha^3}{48}\right)C_A^3+\left(\frac{31}{9}+\alpha\right)N_fT_FC_FC_A \\
&+\left(\frac{481}{54}+\frac{19\alpha}{12}+\frac{\alpha^2}{6}\right)N_fT_FC_A^2-\frac{50}{27}N_f^2T_F^2C_A-\frac{8}{9}N_f^2T_F^2C_F\Bigg] \\
&+\frac{a^3}{\epsilon^3}\Bigg[\left(\frac{403}{144}+\frac{47\alpha}{48}+\frac{\alpha^2}{6}-\frac{\alpha^3}{3}\right)C_A^3 \\
&+\left(-\frac{22}{9}-\frac{5\alpha}{6}-\frac{\alpha^2}{3}\right)N_fT_FC_A^2+\frac{4}{9}N_f^2T_F^2C_A\Bigg]+O(a^4)\ . \qquad (8.59)
\end{aligned}
$$

$$
\begin{aligned}
Z_\psi \;=\;& 1+\frac{a}{\epsilon}(-\alpha C_F)+\frac{a^2}{\epsilon}\left[T_FN_fC_F+\frac{3}{4}C_F^2+\left(-\frac{25}{8}-\alpha-\frac{\alpha^2}{8}\right)C_FC_A\right] \\
&+\frac{a^2}{\epsilon^2}\left[\left(\frac{3\alpha}{4}+\frac{\alpha^2}{4}\right)C_AC_F+\frac{\alpha^2}{2}C_F^2\right] \\
&+\frac{a^3}{\epsilon}\Bigg[\left(-\frac{9155}{432}+\frac{23\zeta(3)}{8}-\frac{263\alpha}{96}-\frac{\zeta(3)\alpha}{4}-\frac{13\alpha^2}{32}-\frac{\zeta(3)\alpha^2}{8}-\frac{5\alpha^3}{48}\right)C_FC_A^2 \\
&+\left(\frac{143}{12}-4\zeta(3)\right)C_F^2C_A-\frac{1}{2}C_F^3+\left(\frac{287}{27}+\frac{17\alpha}{12}\right)N_fT_FC_FC_A \\
&-N_fT_FC_F^2-\frac{20}{27}N_f^2T_F^2C_F\Bigg] \\
&+\frac{a^3}{\epsilon^2}\Bigg[\left(\frac{275}{36}+\frac{73\alpha}{24}+\frac{3\alpha^2}{4}+\frac{\alpha^3}{8}\right)C_FC_A^2-\frac{3\alpha}{4}C_F^3+\frac{8}{9}N_f^2T_F^2C_F \\
&+\left(-\frac{11}{6}+\frac{25\alpha}{8}+\alpha^2+\frac{\alpha^3}{8}\right)C_F^2C_A \\
&+\left(-\frac{47}{9}-\alpha\right)N_fT_FC_FC_A+\left(\frac{2}{3}-\alpha\right)N_fT_FC_F^2\Bigg] \\
&+\frac{a^3}{\epsilon^3}\Bigg[\left(-\frac{31\alpha}{24}-\frac{3\alpha^2}{8}-\frac{\alpha^3}{12}\right)C_FC_A^2-\frac{\alpha^3}{6}C_F^3 \\
&+\left(-\frac{3\alpha^2}{4}-\frac{\alpha^3}{4}\right)C_F^2C_A+\frac{\alpha}{3}N_fT_FC_FC_A\Bigg]+O(a^4)\ . \qquad (8.60)
\end{aligned}
$$

$$
\begin{aligned}
Z_c \;=\;& 1+\frac{a}{\epsilon}\left[\frac{3}{4}-\frac{\alpha}{4}\right]+\frac{a^2}{\epsilon}\left[\left(\frac{95}{96}+\frac{\alpha}{32}\right)C_A^2-\frac{5}{12}C_AT_FN_f\right] \\
&+\frac{a^2}{\epsilon^2}\left[\left(-\frac{35}{32}+\frac{3\alpha^2}{32}\right)C_A^2+\frac{1}{2}C_AT_FN_f\right] \\
&+\frac{a^3}{\epsilon}\Bigg[\left(\frac{15817}{5184}+\frac{\zeta(3)\alpha}{8}+\frac{\zeta(3)\alpha^2}{32}+\frac{3\zeta(3)}{32}-\frac{17\alpha}{96}-\frac{\alpha^2}{32}-\frac{\alpha^3}{64}\right)C_A^3 \\
&+\left(-\frac{15}{4}+4\zeta(3)\right)T_FN_fC_FC_A+\left(-\frac{97}{324}-3\zeta(3)+\frac{7\alpha}{24}\right)T_FN_fC_A^2
\end{aligned}
$$

$$
\begin{aligned}
& - \frac{35}{81} T_F^2 N_f^2 C_A \Big] \\
& + \frac{a^3}{\epsilon^2} \left[\left(- \frac{15587}{3456} + \frac{5\alpha}{96} + \frac{13\alpha^2}{128} + \frac{\alpha^3}{48} \right) C_A^3 \right. \\
& \left. \quad + T_F N_f C_F C_A + \left(\frac{1405}{432} - \frac{\alpha}{48} \right) T_F N_f C_A^2 - \frac{10}{27} T_F^2 N_f^2 C_A \right] \\
& + \frac{a^3}{\epsilon^3} \left[\left(\frac{2765}{1152} + \frac{35\alpha}{384} - \frac{9\alpha^2}{128} - \frac{5\alpha^3}{128} \right) C_A^3 \right. \\
& \left. \quad + \left(- \frac{149}{72} - \frac{\alpha}{24} \right) T_F N_f C_A^2 + \frac{4}{9} T_F^2 N_f^2 C_A \right] + O(a^4) \ . \qquad (8.61)
\end{aligned}
$$

The three loop $\overline{\text{MS}}$ renormalization constants, (8.59), (8.60) and (8.61) all agree with results previously obtained for Quantum Chromodynamics, fixed in an arbitrary linear covariant gauge. The presence of the auxiliary fields and tensor coupling in the massless part of the Lagrangian (8.2) has not affected the three loop renormalization properties.

For the $\langle B^a_{\mu\nu} \bar{B}^b_{\rho\sigma} \rangle$ 2-point function, the full $\overline{\text{MS}}$ renormalization constant necessary to render the corrections finite up to three loops is given by

$$
\begin{aligned}
Z_B \;=\; & 1 + [3 - \alpha] C_A \frac{a}{\epsilon} + \left[-1 - \frac{9\alpha}{4} + \frac{3\alpha^2}{4} \right] C_A^2 \frac{a^2}{\epsilon^2} + 2 C_A T_F N_f \frac{a^2}{\epsilon^2} \\
& + \left[\frac{61}{12} - \alpha - \frac{\alpha^2}{8} \right] C_A^2 \frac{a^2}{\epsilon} + 2 C_A T_F N_f \frac{a^2}{\epsilon} - \frac{1}{256 N_A} \lambda^{abcd} \lambda^{acbd} \frac{1}{\epsilon} \\
& + \left[\left(\frac{13}{9} + \frac{47\alpha}{24} + \frac{9\alpha^2}{8} - \frac{\alpha^3}{2} \right) C_A^3 + \left(- \frac{34}{9} - \frac{5\alpha}{3} \right) T_F N_f C_A^2 + \frac{16}{9} T_F^2 N_f^2 C_A \right] \frac{a^3}{\epsilon^3} \\
& + \left[- \frac{40}{27} N_f^2 T_F^2 C_A + 4 N_f T_F C_F C_A + \left(\frac{277}{27} + \frac{2\alpha}{3} \right) N_f T_F C_A^2 \right. \\
& \left. + \left(- \frac{919}{108} - \frac{121\alpha}{24} + \frac{11\alpha^2}{8} + \frac{\alpha^3}{4} \right) C_A^3 \right] \frac{a^3}{\epsilon^2} \\
& + \left[- \frac{C_A}{32 N_A} f_4^{abcd} \lambda^{acbd} - \frac{1}{8 N_A} f_4^{abcd} f_4^{apcq} \lambda^{bqdp} \right] \frac{a^2}{\epsilon^2} \\
& + \frac{C_A}{256 N_A} (1 + \alpha) \lambda^{abcd} \lambda^{acbd} \frac{a}{\epsilon^2} \\
& - \frac{1}{3072 N_A} \left[3 \lambda^{abcd} \lambda^{acpq} \lambda^{bpdq} + \lambda^{abcd} \lambda^{apcq} \lambda^{bqcp} \right] \frac{1}{\epsilon^2} \\
& + \left[- \frac{140}{81} N_f^2 T_F^2 C_A + (16\zeta(3) - 15) N_f T_F C_F C_A^2 \right. \\
& + \left(\frac{17\alpha}{12} - 16\zeta(3) - \frac{5}{162} \right) N_f T_F C_A^2 \\
& \left. + \left(\frac{18193}{1296} + \frac{9\zeta(3)}{8} - \frac{271\alpha}{96} - \frac{13\alpha^2}{32} - \frac{\zeta(3)\alpha^2}{8} - \frac{5\alpha^3}{48} \right) C_A^3 \right] \frac{a^3}{\epsilon} \\
& + \left[\frac{C_A}{N_A} \left(\frac{\zeta(3)}{8} - \frac{13}{192} \right) f_4^{abcd} \lambda^{acbd} + \frac{1}{N_A} \left(\frac{\zeta(3)}{2} - \frac{13}{48} \right) f_4^{abcd} f_4^{apcq} \lambda^{bqdp} \right] \frac{a^2}{\epsilon} \\
& - \frac{5 C_A}{192 N_A} \lambda^{abcd} \lambda^{acbd} \frac{a}{\epsilon}
\end{aligned}
$$

$$+\frac{1}{6144N_A}\left[3\lambda^{abcd}\lambda^{acpq}\lambda^{bpdq}+\lambda^{abcd}\lambda^{apcq}\lambda^{bqdp}\right]\frac{1}{\epsilon}+O(a^4;\lambda^4) \quad , \tag{8.62}$$

where the tensor

$$f_4^{abcd} = f^{abe}f^{cde} \quad . \tag{8.63}$$

Also, we have used the convention

$$\lambda^{abcd} \hookrightarrow \frac{\lambda^{abcd}}{16\pi^2} \quad . \tag{8.64}$$

We note here that, in accordance with the Ward identity given by the algebraic renormalization procedure, the three loop corrections to the $\langle G^a_{\mu\nu}\bar{G}^b_{\rho\sigma}\rangle$ 2-point function are rendered finite using an identical renormalization constant, $Z_G = Z_B$. An independent verification of this result does not exist in the literature, a strong check on the correctness of the expression (8.62) is made possible by using the multiplicative renormalizability of the Green's function that is described by the vertex,

$$\langle A^a_\mu B^b_{\nu\rho}\bar{B}^c_{\sigma\tau}\rangle = \sqrt{Z_A}\sqrt{Z_B}Z_g\langle A^a_{o\,\mu}B^b_{o\,\nu\rho}\bar{B}^c_{o\,\sigma\tau}\rangle \quad . \tag{8.65}$$

The vertex $A^a_\mu\bar{B}^b_{\nu\rho}B^c_{\sigma\tau}$ is calculated using the MINCER algorithm by making reference to a modified 2-point function in which the external momentum associated with the field $B^c_{\sigma\tau}$ is nullified (set equal to zero). Deriving 2-point functions in this way relies on the fact that there are no infrared divergent factors of the form $1/(k^2)^2$ present in a Feynman integral where k is an internal momentum. Any possibility of such factors is identified by incorporating suitable count statements into the automatic FORM routine before the MINCER algorithm is applied. These safety checks were incorporated into the algorithm used for this calculation where the modified 2-point function used to describe the Green's function (8.65) was declared infrared safe. The poles for this vertex are extracted using the projector

$$\frac{(\delta_{\nu\sigma}\delta_{\rho\tau}-\delta_{\nu\tau}\delta_{\rho\sigma})p_\mu}{d(d-1)p^2} \quad .$$

Calculating this vertex to three loops and applying the appropriate factors of (8.59) and (8.62), all remaining poles in ϵ on the RHS of (8.65) are canceled by the renormalization constant

$$\begin{aligned}
Z_g \;=\; & 1+\frac{a}{\epsilon}\left[-\frac{11}{6}C_A+\frac{2}{3}T_FN_f\right]+\frac{a^2}{\epsilon}\left[-\frac{17}{6}C_A^2+T_FN_fC_F+\frac{5}{3}T_FN_fC_F\right] \\
& +\frac{a^2}{\epsilon^2}\left[\frac{121}{24}C_A^2+\frac{2}{3}T_F^2N_f^2-\frac{11}{3}T_FN_fC_A\right] \\
& +\frac{a^3}{\epsilon}\left[-\frac{2857}{324}C_A^3+\frac{1415}{162}N_fT_FC_A^2\right. \\
& \left.+\frac{205}{54}N_fT_FC_FC_A-\frac{1}{3}N_fT_FC_F^2-\frac{79}{81}N_f^2T_F^2C_A-\frac{22}{27}N_f^2T_F^2C_F\right]
\end{aligned}$$

$$
+\frac{a^3}{\epsilon^2}\left[\frac{2057}{108}C_A^3-\frac{979}{54}N_fT_FC_A^2-\frac{121}{18}N_fT_FC_FC_A\right.
$$
$$
\left.+\frac{110}{27}N_f^2T_F^2C_A+\frac{22}{9}N_f^2T_F^2C_F\right]
$$
$$
+\frac{a^3}{\epsilon^3}\left[-\frac{6655}{432}C_A^3+\frac{605}{36}N_f^2T_FC_A^2-\frac{55}{9}N_f^2T_F^2C_A+\frac{20}{7}N_f^3T_F^3\right]+O(a^4)\,. \tag{8.66}
$$

The expression (8.66) is consistent with previously established results, [12], [13], [72], [73], [75], [76], [77], [71], [78], and represents a strong consistency check on the correctness of the expression (8.62), which is used in the next part of the calculation.

Explicit $\overline{\mathrm{MS}}$ renormalization of the gauge invariant mass operator

$$
\mathcal{O}=\left(B^a_{\mu\nu}-\bar{B}^a_{\mu\nu}\right)F^a_{\mu\nu}\ , \tag{8.67}
$$

considered as a mass insertion, is carried out using the modified 2-point function

$$
\langle A^a_\mu \mathcal{O} B^b_{\nu\rho}\rangle=\sqrt{Z}_A\sqrt{Z}_B Z_{\mathcal{O}}\langle A^a_{o\,\mu}\mathcal{O}_o B^b_{o\,\nu\rho}\rangle\ . \tag{8.68}
$$

Calculating three loop corrections and applying the appropriate factors from expressions (8.59) and (8.62) poles in ϵ on the RHS of the Green's function (8.68) are canceled using the $\overline{\mathrm{MS}}$ renormalization constant

$$
\begin{aligned}
Z_{\mathcal{O}} \;=\;& 1+\left[\frac{2}{3}T_FN_f-\frac{11}{6}C_A\right]\frac{a}{\epsilon}+\left[\frac{121}{24}C_A^2+\frac{2}{3}T_F^2N_f^2-\frac{11}{3}T_FN_fC_A\right]\frac{a^2}{\epsilon^2}\\
&+\left[\left(\frac{1}{3}T_FN_fC_A-\frac{77}{48}C_A^2+T_FN_fC_F\right)a^2+\frac{1}{512N_A}\lambda^{abcd}\lambda^{acbd}\right.\\
&\left.\quad+\frac{1}{32N_A}f_4^{abcd}a\lambda^{acbd}\right]\frac{1}{\epsilon}\\
&+\left[\frac{605}{36}T_FN_fC_A^2-\frac{6655}{432}C_A^3-\frac{55}{9}T_F^2N_f^2C_A+\frac{20}{27}T_F^3N_f^3\right]\frac{a^3}{\epsilon^3}\\
&+\left[\left(\frac{3989}{288}C_A^3-\frac{757}{72}T_FN_fC_A^2-\frac{121}{18}T_FN_fC_AC_F\right.\right.\\
&\quad\left.+2T_F^2N_f^2C_A+\frac{22}{9}T_F^2N_f^2C_F\right)a^3\\
&\quad+\frac{1}{6144N_A}\left(3\lambda^{abcd}\lambda^{acpq}\lambda^{bpdq}+\lambda^{abcd}\lambda^{apcq}\lambda^{bqdp}\right)\\
&\quad+\frac{1}{N_A}\left(\frac{1}{384}f_4^{abcd}\lambda^{acpq}\lambda^{bdpq}+\frac{1}{256}f_4^{abcd}\lambda^{apbq}\lambda^{cpdq}+\frac{1}{384}f_4^{abcd}\lambda^{apcq}\lambda^{bpdq}\right)a\\
&\quad-\frac{C_A}{N_A}\left(\frac{1}{384}\lambda^{abcd}\lambda^{abcd}+\frac{31}{3072}\lambda^{abcd}\lambda^{acbd}\right)a+\frac{T_FN_f}{768N_A}\lambda^{abcd}\lambda^{acbd}a\\
&\quad\left.+\frac{1}{16N_A}f_4^{abcd}f_4^{apcq}\lambda^{bqdp}a^2-\frac{41C_A}{288N_A}f_4^{abcd}\lambda^{acbd}a^2+\frac{5T_FN_f}{144N_A}f_4^{abcd}\lambda^{acbd}a^2\right]\frac{1}{\epsilon^2}\\
&+\left[\left(\frac{211}{108}T_FN_fC_A^2-\frac{361}{96}C_A^3+\frac{97}{54}T_FN_fC_AC_F-\frac{5}{27}T_F^2N_f^2C_A-\frac{22}{27}T_F^2N_f^2C_F\right.\right.\\
&\quad\left.-\frac{1}{3}T_FN_fC_F^2\right)a^3-\frac{1}{12288N_A}\left(3\lambda^{abcd}\lambda^{acpq}\lambda^{bpdq}+\lambda^{abcd}\lambda^{apcq}\lambda^{bqdp}\right)
\end{aligned}
$$

$$+ \frac{1}{N_A}\left(\frac{5}{128}f_4^{abcd}\lambda^{acpq}\lambda^{bdpq} - \frac{31}{384}f_4^{abcd}\lambda^{acpq}\lambda^{bpdq} + \frac{17}{512}f_4^{abcd}\lambda^{apbq}\lambda^{cpdq} + \frac{5}{128}f_4^{abcd}\lambda^{apcq}\lambda^{bpdq}\right) a$$

$$+ \frac{C_A}{N_A}\left(\frac{79}{1536}\lambda^{abcd}\lambda^{acbd} - \frac{5}{128}\lambda^{abcd}\lambda^{abcd}\right) a - \frac{19}{96N_A}f_4^{abcd}f_4^{apcq}\lambda^{bqdp}a^2$$

$$+ \left.\frac{515C_A}{3456N_A}f_4^{abcd}\lambda^{acbd}a^2 + \frac{T_F N_f}{432N_A}f_4^{abcd}\lambda^{acbd}a^2\right]\frac{1}{\epsilon} + O(a^4;\lambda^4) \ . \quad (8.69)$$

The gauge invariant mass operator (8.1) has been inserted into the massless part of the Lagrangian (8.2) which we have quantized in an arbitrary linear covariant gauge. The important thing to notice about the expression (8.69) is that it is independent of the gauge parameter α. Again we have satisfied a strong consistency check on the correctness of our calculation.

8.7 Renormalization group functions

For our theory, fixed in an arbitrary linear covariant gauge, with gauge parameter α, the anomalous dimensions for the fields is given by

$$\gamma_\phi(a,\lambda,\alpha) = \frac{1}{Z_\phi}\left[\beta(a)\frac{\partial Z_\phi}{\partial a} + \beta_\lambda^{pqrs}(a,\lambda)\frac{\partial Z_\phi}{\partial \lambda^{pqrs}} - \alpha\gamma_\alpha(a,\alpha)\frac{\partial Z_\phi}{\partial \alpha}\right] , \quad (8.70)$$

The quartic tensor coupling β-function, β_λ^{pqrs}, calculated to one loop in [59] was discussed earlier. The QCD β-function is derived by observing that, to a given loop order, the bare and renormalized Yang-Mills coupling, g_o and g, are related by the expression

$$g_o = \mu^\epsilon g Z_g \ , \quad (8.71)$$

where, for our purposes, μ describes the dimensionful parameter of the dimensional regularization process used to maintain a dimensionless coupling constant in d-dimensions. The bare, unrenormalized, Yang-Mills coupling is independent of μ, and so $\beta(a)$ is obtained by differentiating both sides of (8.71) with respect to $\mu\frac{\mathrm{d}}{\mathrm{d}\mu}$ such that

$$\begin{aligned} 0 = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\left(\mu^\epsilon g Z_g\right) &= \epsilon\mu^\epsilon g Z_g + \mu^\epsilon\mu\frac{\partial g}{\partial\mu}\frac{\partial}{\partial g}\left(gZ_g\right) \\ 0 &= \epsilon g Z_g + \beta(g)\frac{\partial}{\partial g}\left(gZ_g\right) \ . \end{aligned} \quad (8.72)$$

By defining the QCD β-function using the relation

$$\beta(g) = \mu\frac{\partial g}{\partial \mu} \ , \quad (8.73)$$

it is elementary to show that

$$\beta(g) = -\epsilon g Z_g\left[\frac{\partial}{\partial g}\left(gZ_g\right)\right]^{-1} \ . \quad (8.74)$$

As such, it is straightforward to implement (8.74) into a computer algebra program, such as MAPLE or REDUCE, using the expression (8.66) and deduce that the three loop QCD β-function is given by

$$
\begin{aligned}
\beta(g) &= \frac{(d-4)}{2} - \left[\frac{11}{3}C_A - \frac{4}{3}T_F N_f\right] a - \left[\frac{34}{3}C_A^2 - \frac{20}{3}C_A T_F N_f - 4T_F N_f C_F\right] a^2 \\
& - \left[\frac{2857}{54}C_A^3 - \frac{1415}{27}C_A^2 T_F N_f + \frac{158}{27}C_A T_F^2 N_f^2 - \frac{205}{9}C_A T_F N_f C_F + \right. \\
& \left. \frac{44}{9}T_F^2 N_f^2 C_F + 2T_F N_f C_F^2\right] a^3 + O(a^4) \ .
\end{aligned} \tag{8.75}
$$

This value is consistent with previously established results, [12], [13], [72], [73], [75], [76], [77], [71], [78]. For a theory fixed in an arbitrary linear covariant gauge, non-zero values for the gauge parameter, α, also undergo renormalization, such that

$$
\alpha = Z_\alpha \alpha_o \ . \tag{8.76}
$$

The gauge parameter undergoes a renormalization analogous to a composite gluon field such that, $Z_\alpha = Z_A$. As such the anomalous dimension for the gauge parameter is closely related to the anomalous dimension for the gluon field. We will discuss how γ_α enters into the expression for the gluon field anomalous dimension, γ_A, and demonstrate how the renormalization group functions represent a valuable consistency check on the residual poles in, $1/\epsilon^2, 1/\epsilon^3, \cdots$, calculated for higher loop order corrections by making reference to a specific two loop example in Yang-Mills theory. It is instructive to begin a consideration of the renormalization group equations by showing how, in the Landau gauge, the one-loop QCD β-function dictates the correct form that must be taken by the residual pole, $\propto a^2/\epsilon^2$, resulting from the two-loop correction to the gluon propagator. Recalling the explicit form of the two-loop expression for Z_A in the Landau gauge,

$$
\begin{aligned}
Z_A|_{\alpha=0} &= 1 + \frac{z_{A11}a}{\epsilon} + \frac{z_{A21}a^2}{\epsilon} + \frac{z_{A22}a^2}{\epsilon^2} + O(a^3) \\
&= 1 + \frac{a}{\epsilon}\left[\frac{13}{6}C_A - \frac{4}{3}T_F N_f\right] \\
& + \frac{a^2}{\epsilon}\left[\frac{59}{16}C_A^2 - \frac{5}{2}C_A T_F N_f - 2C_F T_F N_f\right] \\
& + \frac{a^2}{\epsilon^2}\left[-\frac{13}{8}C_A^2 + C_A T_F N_f\right] + O(a^3) \ ,
\end{aligned} \tag{8.77}
$$

and the one-loop expression for the QCD β-function,

$$
\begin{aligned}
\beta(a) &= \frac{(d-4)}{2} + b_{a1}a \\
&= -\epsilon - \left[\frac{11}{3}C_A - \frac{4}{3}T_F N_f\right] a \ .
\end{aligned} \tag{8.78}
$$

Given this, the explicit expression for the anomalous dimension of the gluon, indeed any QCD field is in fact very simple and given by,

$$
\gamma_A(a,\lambda) = -\sum_1^n n z_{An1} a^n \ . \tag{8.79}
$$

At one-loop order, only the infinitesimal piece of $\beta(a)$ is included and the renormalization group functions are a trivial re-statement of the renormalization constant expressed as a finite quantity. At two-loop order and above, residual pole terms calculated using Feynman diagrams must satisfy a consistency condition dictated by the renormalization group function and expressed using results from the previous order in perturbation theory. Described using pole and β-function coefficients, the two-loop Landau gauge anomalous dimension for the gluon field is given by,

$$\gamma_A(a) \;=\; \frac{a^2}{\epsilon}\left(z_{A11}^2 + z_{A11}c_{a1} - 2z_{A22}\right) - az_{A11} - 2a^2 z_{A22} + O(a^3) \;, \qquad (8.80)$$

where it is straightforward to see that in order to have a finite expression for $\gamma_A(a)$, we require that,

$$z_{A22} \;=\; \frac{z_{A11}}{2}\left(z_{A11} + b_{a1}\right) \;. \qquad (8.81)$$

For arbitrary linear covariant gauge, the anomalous dimension for non-vanishing values of the gauge parameter, alpha, is identical to that for the composite gluon field, or indeed just the gluon field, That is, to one-loop order,

$$\gamma_\alpha(a,\alpha) \;=\; g_{\alpha 1}a + O(a^2) \;. \qquad (8.82)$$

In an arbitrary linear covariant gauge, the two-loop anomalous dimension for the gluon field, described using pole, β-function and gauge parameter anomalous dimension coefficients, is now given,

$$\begin{aligned}\gamma_A(a) \;=\;& \frac{a^2}{\epsilon}\left(z_{A11}^2 + z_{A11}b_{a1} + \alpha g_{\alpha 1}\frac{\partial z_{A11}}{\partial\alpha} - 2z_{A22}\right)\\ & -az_{A11} - 2a^2 z_{A22} + O(a^3) \;. \end{aligned}\qquad (8.83)$$

Here we see that, for non-vanishing α, the one loop restrictions imposed on the two-loop residual pole of the gluon renormalization constant include the anomalous dimension for the gauge parameter. The functions of the renormalization group equation simply express the fact that bare, unrenormalized, objects, ϕ_o, have no dependence on the renormalized expressions, a_r and α_r, used to express their renormalized counterparts, ϕ_r. To a loop practitioner, they represent an invaluable check on manipulations with sums of Feynman diagrams that become more complicated at increasing orders in perturbation theory.

In the absence of an explicit tensor structure it is elementary, and at three loop order desirable, to calculate the anomalous dimension by defining the individual elements of (8.70) in a computer algebra package such as REDUCE or MAPLE and then expanding to the required order in the couplings. For the ordinary QCD objects this is what we did, leading to the results,

$$\gamma_A(a) \;=\; \left[(3\alpha - 13)C_A + 8T_F N_f\right]\frac{a}{6}$$

$$
\begin{aligned}
&+\left[(2\alpha^2+11\alpha-59)C_A^2+40C_AT_FN_f+32C_FT_FN_f\right]\frac{a^2}{8}\\
&+\Big[\Big(63\alpha^3+297\alpha^2+1503\alpha-9965\quad+(54\alpha^2+216\alpha+162)\zeta(3)\Big)\,C_A^3\\
&\quad+(-576\alpha+14576-5184\zeta(3))\,C_A^2T_FN_f+\Big(6912\zeta(3)+80\Big)C_AC_FT_FN_f\\
&\quad-2432C_AT_F^2N_f^2-576C_F^2T_FN_f-1408C_FT_F^2N_f^2\Big]\frac{a^3}{288}+O(a^4) \qquad (8.84)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_c(a) &= (\alpha-3)C_A\frac{a}{4}\\
&+\left[-(3\alpha+95)C_A^2+40C_AT_FN_f\right]\frac{a^2}{48}\\
&+\Big[\Big(81\alpha^3-162\alpha^2\zeta(3)+162\alpha^2-648\alpha\zeta(3)+918\alpha-486\zeta(3)-15817\Big)\,C_A^3\\
&\quad+(-1512\alpha+15552\zeta(3)+1552)\,C_A^2T_FN_f\\
&\quad+(-20736\zeta(3)+19440)\,C_AC_FT_FN_f+2240C_AT_f^2N_f^2\Big]\frac{a^3}{1728}+O(a^4) \qquad (8.85)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_\psi(a) &= \alpha C_F a\\
&+\Big[\Big(\alpha^2+8\alpha+25\Big)\,C_AC_F-6C_F^2-8C_FT_FN_f\Big]\frac{a^2}{4}\\
&+\Big[\Big(90\alpha^3+108\zeta(3)\alpha^2+351\alpha^2+216\zeta(3)\alpha+2367\alpha-2484\zeta(3)\\
&\quad+18310\Big)\,C_A^2C_F-(1224\alpha+9184)\,C_AC_FT_FN_f+423C_F^3+864C_F^2T_fN_f\\
&\quad+(3456\zeta(3)-10296)\,C_AC_F^2+640C_FT_F^2N_f^2\Big]\frac{a^3}{288}+O(a^4) \qquad (8.86)
\end{aligned}
$$

The anomalous dimensions for the localization fields and mass operator objects, $\phi \in \{B^a_{\mu\nu}, G^a_{\mu\nu}, \mathcal{O}\}$, are derived using renormalization constants with an explicit dependence on the quartic tensor coupling. It is not possible to evaluate such functions using computer algebra programs. Recalling the symmetry properties obeyed by the tensor,

$$
\begin{aligned}
\lambda^{abcd} &= \lambda^{cdab}\\
\lambda^{abcd} &= \lambda^{bacd}\ , \qquad (8.87)
\end{aligned}
$$

we observe that the quartic tensor coupling is differentiated using the formula,

$$
\begin{aligned}
\frac{\partial\lambda^{abcd}}{\partial\lambda^{pqrs}} &= \frac{1}{8}\Big[\ \delta^{ap}\delta^{bq}\delta^{cr}\delta^{ds}+\delta^{aq}\delta^{bp}\delta^{cr}\delta^{ds}\\
&+\delta^{ap}\delta^{bq}\delta^{cs}\delta^{dr}+\delta^{aq}\delta^{bp}\delta^{cs}\delta^{dr}\\
&+\delta^{cp}\delta^{dq}\delta^{ar}\delta^{bs}+\delta^{cq}\delta^{dp}\delta^{ar}\delta^{bs}\\
&+\delta^{cp}\delta^{dq}\delta^{as}\delta^{br}+\delta^{cq}\delta^{dp}\delta^{as}\delta^{br}\ \Big]\ . \qquad (8.88)
\end{aligned}
$$

As might be expected, differentiating quartic tensor couplings in the carefully ordered renormalization constants destroys the structures that were introduced by the group theory algorithm. For example, using the product rule and (8.88) to differentiate a structure including three quartic tensor couplings,

$$\begin{aligned}
&\frac{\partial}{\partial\lambda^{b_1b_2b_3b_4}}\left(\lambda^{a_1a_2a_3a_4}\lambda^{a_1a_3a_5a_6}\lambda^{a_2a_5a_4a_6}\right)\\
= \quad & \frac{1}{8}\lambda^{a_1a_3b_1b_3}\lambda^{a_1b_2a_3b_4} + \frac{1}{8}\lambda^{a_1a_3b_1b_3}\lambda^{a_1b_4a_3b_2} + \frac{1}{8}\lambda^{a_1a_3b_1b_4}\lambda^{a_1b_2a_3b_3}\\
&+\frac{1}{8}\lambda^{a_1a_3b_1b_4}\lambda^{a_1b_3a_3b_2} + \frac{1}{8}\lambda^{a_1a_3b_2b_3}\lambda^{a_1b_1a_3b_4} + \frac{1}{8}\lambda^{a_1a_3b_2b_3}\lambda^{a_1b_4a_3b_1}\\
&+\frac{1}{8}\lambda^{a_1a_3b_2b_4}\lambda^{a_1b_1a_3b_3} + \frac{1}{8}\lambda^{a_1a_3b_2b_4}\lambda^{a_1b_3a_3b_1} + \frac{1}{4}\lambda^{a_2b_1a_4b_2}\lambda^{a_2b_3a_4b_4}\\
&+\frac{1}{4}\lambda^{a_2b_1a_4b_2}\lambda^{a_2b_4a_4b_3} + \frac{1}{4}\lambda^{a_2b_2a_4b_1}\lambda^{a_2b_3a_4b_4} + \frac{1}{4}\lambda^{a_2b_2a_4b_1}\lambda^{a_2b_4a_4b_3}\\
&+\frac{1}{8}\lambda^{a_5a_6b_1b_3}\lambda^{a_5b_2a_6b_4} + \frac{1}{8}\lambda^{a_5a_6b_1b_3}\lambda^{a_5b_4a_6b_2} + \frac{1}{8}\lambda^{a_5a_6b_1b_4}\lambda^{a_5b_2a_6b_3}\\
&+\frac{1}{8}\lambda^{a_5a_6b_1b_4}\lambda^{a_5b_3a_6b_2} + \frac{1}{8}\lambda^{a_5a_6b_2b_3}\lambda^{a_5b_1a_6b_4} + \frac{1}{8}\lambda^{a_5a_6b_2b_3}\lambda^{a_5b_4a_6b_1}\\
&+\frac{1}{8}\lambda^{a_5a_6b_2b_4}\lambda^{a_5b_1a_6b_3} + \frac{1}{8}\lambda^{a_5a_6b_2b_4}\lambda^{a_5b_3a_6b_1}\,, \qquad (8.89)
\end{aligned}$$

results in a particularly long winded expression. Applying the tensor differentiation to all of the quartic tensor couplings in the three loop renormalization constants has the result that raw data used to construct the renormalization group functions is given by disordered products of tensor functions. To restore order, it is necessary to include a further group theory algorithm incorporating a further re-ordering and re-labelling process and more, carefully chosen, combinations of the Jacobi identities, (8.21) and (8.33). To be sure that the initial expressions for $\gamma_B(a,\lambda)$ and $\gamma_{\mathcal{O}}(a,\lambda)$ do not become too complicated, it is important to give a group theory treatment to each component used to construct them before these are input into the formula (8.70). Fortunately, for the most difficult factors $\frac{1}{Z_\phi}$, $\phi \in \{B^a_{\mu\nu}, G^a_{\mu\nu}, \mathcal{O}\}$, it is necessary to consider only the two-loop renormalization constants. Given this, the factors $\frac{1}{Z_\phi}$ are expanded using the two-loop renormalization constants in a symbolic manipulation program (FORM) around,

$$a = \lambda = 0 \qquad (8.90)$$

to $O(a^3;\lambda^3)$. This is done using a double Taylor expansion given by,

$$\begin{aligned}
\frac{1}{Z_\phi} \quad = \quad & 1 - \left.\frac{\partial Z_\phi}{\partial a}\right|_{a=\lambda=0} a - \left.\frac{\partial Z_\phi}{\partial\lambda^{pqrs}}\right|_{a=\lambda=0}\lambda^{pqrs}\\
&+\frac{1}{2}\left(\left.\frac{\partial Z_\phi}{\partial a}\frac{\partial Z^2_\phi}{\partial a}\right|_{a=\lambda=0} - \left.\frac{\partial^2 Z_\phi}{\partial a^2}\right|_{a=\lambda=0}\right)a^2\\
&+\frac{1}{2}\left(\left.\frac{\partial Z_\phi}{\partial\lambda^{pqrs}}\frac{\partial Z^2_\phi}{\partial\lambda^{p'q'r's'}}\right|_{a=\lambda=0} - \left.\frac{\partial^2 Z_\phi}{\partial\lambda^{pqrs}\partial p'q'r's'}\right|_{a=\lambda=0}\right)\lambda^{pqrs}\lambda^{p'q'r's'}\\
&+\left(\left.\frac{\partial Z_\phi}{\partial\lambda^{pqrs}}\frac{\partial Z^2_\phi}{\partial a}\right|_{a=\lambda=0} - \left.\frac{\partial^2 Z_\phi}{\partial a\partial\lambda^{pqrs}}\right|_{a=\lambda=0}\right)a\lambda^{pqrs}\,. \qquad (8.91)
\end{aligned}$$

Each element of (8.91) was computed individually and inserted into the expressions for $\frac{1}{Z_\phi}$ using a short FORM script. The terms differentiated by the couplings in (8.70) are computed using the full three-loop renormalization constants. The remaining elements of (8.70) are the one loop β-function for the quartic tensor coupling, (8.58), and the two loop piece of the QCD beta function, (8.75). As individual elements of (8.70) are given in terms of tensor expressions, when multiplying three components it is important to avoid the use of repeated summed indices. Noting this, it was necessary to use three different index labels,

$$a_1, \cdots, a_4 \ , \ b_1, \cdots, b_4 \text{ and } c_1, \cdots, c_4 \quad , \tag{8.92}$$

to ensure that no repetitions occurred. After all of the different elements have been combined, it is necessary to include a further wholesale re-labelling operation in a final group theory algorithm. From the above it should be clear that, unlike for the case with scalar couplings, for any theory with a tensor coupling deriving the correct form of the renormalization group functions can become very labour intensive.

Before giving the final expressions for the operator object anomalous dimensions, we show the full detail of the symbolic manipulation output for the mass operator anomalous dimension, $\gamma_{\mathcal{O}}$. In this output we have attached labels to all of the components in (8.70), `zop1,zop21,zop22,zop31,zop32,zop33`, identifying which parts come from the simple double and triple poles appearing in the three loop renormalization constant, and `gl0,gl1,ga0,ga1,ga2`, for the infinitesimal, one and two-loop pieces of the β-functions for the quartic tensor and Yang-Mills couplings respectively. Expressing the output in this way shows how results from lower orders in perturbation theory dictate the form that must be taken by the residual poles occurring at higher orders in perturbation theory, in great detail and to three-loop order.

```
gammaop =
  + ep^-2*a^3*Ca^3 * (
      + 6655/144*zop33*ga0
      - 1331/36*zop22*ga1
      - 1331/48*zop1*zop22*ga0
      + 1331/108*zop1^2*ga1
      + 1331/216*zop1^3*ga0
       )
  + ep^-2*a^3*Ca^3 * (
      - 605/512*zo33*ga0
      + 121/3*zo22*ga1
      + 121/4*zo1*zo22*ga0
      + 121/9*zo1^2*ga1
      - 121/18*zo1^3*ga0
```

```
      )
 + ep^-2*a^3*Nf^2*Tf^2*Ca * (
    + 55/3*zo33*ga0
    - 44/3*zo22*ga1
    - 11*zo1*zo22*ga0
    + 44/9*zo1^2*ga1
    + 22/9*zo1^3*ga0
      )
 + ep^-2*a^3*Nf^3*Tf^3 * (
    - 20/9*zo33*ga0
    + 16/9*zo22*ga1
    + 4/3*zo1*zo22*ga0
    - 16/27*zo1^2*ga1
    - 8/27*zo1^3*ga0
      )
 + ep^-1*Nca^-1 * (
    - 1/1024*lambdar(As1,As2,As3,As4)*lambdar(As1,As3,As5,As6)
            *lambdar(As2,As5,As4,As6)*zo32*gl0
    + 1/1024*lambdar(As1,As2,As3,As4)*lambdar(As1,As3,As5,As6)
            *lambdar(As2,As5,As4,As6)*zo21*gl1
    - 1/2048*lambdar(As1,As2,As3,As4)*lambdar(As1,As5,As2,As6)
            *lambdar(As2,As5,As4,As6)*zo32*gl0
    + 1/2048*lambdar(As1,As2,As3,As4)*lambdar(As1,As5,As2,As6)
            *lambdar(As2,As5,As4,As6)*zo21*gl1
    - 1/2048*lambdar(As1,As2,As3,As4)*lambdar(As1,As5,As3,As6)
            *lambdar(As2,As6,As4,As5)*zo32*gl0
    + 1/2048*lambdar(As1,As2,As3,As4)*lambdar(As1,As5,As3,As6)
            *lambdar(As2,As6,As4,As5)*zo21*gl1
      )
 + ep^-1*a*Nca^-1 * (
    - 1/96*ff4(As1,As2,As3,As4)*lambdar(As1,As3,As5,As6)
            *lambdar(As2,As5,As4,As6)*zo32*gl0
    - 1/192*ff4(As1,As2,As3,As4)*lambdar(As1,As3,As5,As6)
```

```
        *lambdar(As2,As5,As4,As6)*zo32*ga0
    + 1/164*ff4(As1,As2,As3,As4)*lambdar(As1,As3,As5,As6)
        *lambdar(As2,As5,As4,As6)*zo21*gl1
    - 1/384*ff4(As1,As2,As3,As4)*lambdar(As1,As5,As2,As6)
        *lambdar(As3,As5,As4,As6)*zo32*gl0
    - 1/768*ff4(As1,As2,As3,As4)*lambdar(As1,As5,As2,As6)
        *lambdar(As3,As5,As4,As6)*zo32*ga1
    + 1/256*ff4(As1,As2,As3,As4)*lambdar(As1,As5,As2,As6)
        *lambdar(As3,As5,As4,As6)*zo21*gl1
  )
ep^-1*a*Nca^*Ca * (
    + 13/512*lambdar(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
        *zo32*gl0
    + 13/1024*lambdar(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
        *zo32*ga0
    - 7/256*lambdar(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
        *zo21*gl1
    - 11/1536*lambdar(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
        *zo1*zo21*gl0
    - 11/3072*lambdar(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
        *zo1*zo21*ga0
  )
ep^-1*a*Nca^*Tf * (
    - 1/384*lambdar(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
        *zo32*gl0
    - 1/768*lambdar(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
        *zo32*ga0
    + 1/384*lambdar(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
        *zo1*zo21*gl0
    + 1/768*lambdar(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
        *zo1*zo21*ga0
  )
+ ep^-1*a^2*Nca^-1 * (
```

```
    - 1/16*ff4(As1,As2,As3,As4)*ff4(As1,As5,As3,As6)
            *lambdar(As2,As5,As4,As6)*zo32*gl0
    - 1/8*ff4(As1,As2,As3,As4)*ff4(As1,As5,As3,As6)
            *lambdar(As2,As5,As4,As6)*zo32*ga0
    + 3/16*ff4(As1,As2,As3,As4)*ff4(As1,As5,As3,As6)
            *lambdar(As2,As5,As4,As6)*zo21*gl1
     )
+ ep^-1*a*Nca^-1*Ca * (
    + 25/144*ff4(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
            *zo32*gl0
    + 25/72*ff4(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
            *zo32*ga0
    - 15/64*ff4(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
            *zo21*gl1
    - 11/96*ff4(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
            *zo21*ga1
    - 11/192*ff4(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
            *zo1**zo21*gl0
    - 11/96*ff4(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
            *zo1*zo21*gl0
     )
+ ep^-1*a^2*Ca^2 * (
    - 121/12*zo22*ga0
    + 121/18*zo1*ga0
    + 121/36*zo1^2*ga0
     )
+ ep^-1*a^2*Nf*Nca^-1*Tf * (
    - 5/144*ff4(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
            *zo32*gl0
    - 5/72*ff4(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
            *zo32*ga0
    + 1/24*ff4(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
            *zo21*ga1
```

```
     + 1/48*ff4(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
             *zo1*zo21*gl0
     + 1/24*ff4(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
             *zo1*zo21*ga0
      )
 + ep^-1*a^2*Nf*Tf*Ca * (
     + 22/3*zo22*ga0
     + 44/9*zo1*ga1
     - 22/9*zo1^2*ga0
      )
 + ep^-1*a^2*Nf^2*Tf^2 * (
     - 4/3*zo22*ga0
     + 8/9*zo1*ga1
     + 4/9*zo1^2*ga0
      )
 + ep^-1*a^3*Ca^3 * (
     - 3989/96*zo32*ga0
     + 3/16*zo21*gl1
     + 847/72*zo21*ga1
     + 187/9*zo1*ga2
     + 847/96*zo1*zo21*ga0
      )
 + ep^-1*a^3*Nf*Tf*Ca^2 * (
     + 757/24*zo32*ga0
     - 121/18*zo21*ga1
     - 178/9*zo1*ga2
     - 121/24*zo1*zo21*ga0
      )
 + ep^-1*a^3*Nf*Tf*Cf*Ca * (
     + 121/6*zo32*ga0
     - 22/3*zo21*ga1
     - 22/3*zo1*ga2
     - 11/2*zo1*zo21*ga0
```

```
      )
 + ep^-1*a^3*Nf^2*Tf^2*Ca * (
     - 6*zo32*ga0
     + 8/9*zo21*ga1
     + 40/9*zo1*ga2
     + 2/3*zo1*zo21*ga0
      )
 + ep^-1*a^3*Nf^2*Tf^2*Cf * (
     - 22/3*zo32*ga0
     + 8/3*zo21*ga1
     + 8/3*zo1*ga2
     + 2*zo1*zo21*ga0
      )
 + Nca^-1 * (
     + 1/256*lambdar(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
             *zo21*gl0
     + 1/2048*lambdar(As1,As2,As3,As4)*lambdar(As1,As3,As5,As6)
              *lambdar(As2,As5,As4,As6)*zo31*gl0
     + 1/4096*lambdar(As1,As2,As3,As4)*lambdar(As1,As5,As2,As6)
              *lambdar(As3,As5,As4,As6)*zo31*gl0
     + 1/2048*lambdar(As1,As2,As3,As4)*lambdar(As1,As5,As3,As6)
              *lambdar(As2,As6,As4,As5)*zo31*gl0
      )
 + a*Nca^-1 * (
     - 1/32*ff4(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
             *zo21*gl0
     - 1/32*ff4(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
             *zo21*ga0
     + 9/256*ff4(As1,As2,As3,As4)*lambdar(As1,As3,As5,As6)
              *lambdar(As2,As5,As4,As6)*zo31*gl0
     + 1/384*ff4(As1,As2,As3,As4)*lambdar(As1,As3,As5,As6)
              *lambdar(As2,As5,As4,As6)*zo31*ga0
     + 3/256*ff4(As1,As2,As3,As4)*lambdar(As1,As5,As2,As6)
```

```
        *lambdar(As3,As5,As4,As6)*zo31*gl0
  + 3/512*ff4(As1,As2,As3,As4)*lambdar(As1,As5,As2,As6)
        *lambdar(As3,As5,As4,As6)*zo31*ga0
  + 31/768*ff4(As1,As2,As3,As4)*lambdar(As1,As5,As3,As6)
        *lambdar(As2,As4,As5,As6)*zo31*gl0
  )
+ a*Nca^-1*Ca * (
  - 19/768*lambdar(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
      *zo31*gl0
  - 19/1536*lambdar(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
      *zo31*ga0
  )
+ a*Ca * (
  + 11/6*zo1*ga0
  )
+ a*Nf*Tf * (
  - 2/3*zo1*ga0
  )
+ a^2*Nca^-1 * (
  + 19/96*ff4(As1,As2,As3,As4)*ff4(As1,As5,As3,As6)
        *lambdar(As2,As5,As4,As6)*zo31*gl0
  + 19/48*ff4(As1,As2,As3,As4)*ff4(As1,As5,As3,As6)
        *lambdar(As2,As5,As4,As6)*zo31*ga0
  )
+ a^2*Nca^-1*Ca * (
  - 857/3456*ff4(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
      *zo31*gl0
  - 857/1728*ff4(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
      *zo31*ga0
  )
+ a^2*Ca^2 * (
  + 77/24*zo21*ga0
  )
```

```
+ a^2*Nf*Nca^-1*Tf * (
    - 1/432*ff4(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
        *zo31*gl0
    - 1/216*ff4(As1,As2,As3,As4)*lambdar(As1,As3,As2,As4)
        *zo31*ga0
    )
+ a^2*Tf*Ca * (
    - 2/3*zo21*ga0
    )
+ a^2*Tf*Cf * (
    - 2*zo21*ga0
    )
+ a^3*Ca^3 * (
    + 361/32*zo31*ga0
    )
+ a^3*Nf*Tf*Ca^2 * (
    - 211/36*zo31*ga0
    )
+ a^3*Nf*Tf*Cf*Ca * (
    - 97/18*zo31*ga0
    )
+ a^3*Nf*Tf*Cf^2 * (
    + zo31*ga0
    )
+ a^3*Nf^2*Tf^2*Ca * (
    + 5/9*zo31*ga0
    )
+ a^3*Nf^2*Tf^2*Cf * (
    + 22/9*zo31*ga0
    ),                                                    (8.93)
```

where `lambdar` denotes the renormalized tensor coupling and all other factors have been introduced earlier. When the labels `z` and `g` are set equal to one, all divergent expressions cancel and we arrive at the final form of the, finite, anomalous dimensions,

$$
\begin{aligned}
\gamma_B(a,\lambda,\alpha) &= \gamma_G(a,\lambda,\alpha) \\
&= (\alpha-3)C_A + \left[\left(\frac{1}{4}\alpha^2 + 2\alpha - \frac{61}{6}\right)C_A^2 + \frac{10}{3}T_FN_fC_A\right]a^2 \\
&+ \frac{1}{128N_A}\lambda^{abcd}\lambda^{acbd} \\
&+ \left[\left(\frac{5}{16}\alpha^3 + \frac{39}{32}\alpha^2 + \frac{271}{32}\alpha - \frac{18193}{432} + \left(\frac{3}{8}\alpha^2 - \frac{27}{8}\right)\zeta(3)\right)C_A^3\right. \\
&\quad + \left(\frac{5}{54} + 48\zeta(3) - \frac{17}{4}\alpha\right)T_FN_fC_A^2 + (45 - 48\zeta(3))\,T_FN_fC_FC_A \\
&\quad \left. + \frac{140}{27}T_F^2N_f^2C_A\right]a^3 + \left[\frac{3}{8}\zeta(3) - \frac{13}{64}\right]\frac{C_A}{N_A}f_4^{abcd}\lambda^{acbd}a^2 \\
&+ \frac{1}{N_A}\left[\frac{13}{16} - \frac{3}{2}\zeta(3)\right]f_4^{abcd}f_4^{apcq}\lambda^{bpdq}a^2 + \frac{5C_A}{64N_A}\lambda^{abcd}\lambda^{acbd}a \\
&- \frac{1}{2048N_A}\left[3\lambda^{abcd}\lambda^{acpq}\lambda^{bpdq} + \lambda^{abcd}\lambda^{apcq}\lambda^{bqdp}\right] + O(a^4;\lambda^4) \, , \qquad (8.94)
\end{aligned}
$$

and for the gauge, α, independent mass operator,

$$
\begin{aligned}
\gamma_{\mathcal{O}}(a,\lambda) &= \left[\frac{11}{6}C_A - \frac{2}{3}T_FN_f\right]a + \left[\frac{77}{24}C_A^2 - \frac{2}{3}T_FN_fC_A - 2T_FN_fC_F\right]a^2 \\
&- \frac{1}{16N_A}f_4^{abcd}\lambda^{acbd}a - \frac{1}{256N_A}\lambda^{abcd}\lambda^{acbd} \\
&+ \left[\frac{361}{32}C_A^3 - \frac{211}{36}T_FN_fC_A^2 - \frac{97}{18}T_FN_fC_FC_A + T_FN_fC_F^2\right. \\
&\quad \left. + \frac{5}{9}T_F^2N_f^2C_A + \frac{22}{9}T_F^2N_f^2C_F\right]a^3 + \frac{19}{32N_A}f_4^{abcd}f_4^{apcq}\lambda^{bpdq}a^2 \\
&- \frac{1}{N_A}\left[\frac{1}{144}T_FN_f - \frac{857}{1152}C_A\right]f_4^{abcd}\lambda^{acbd}a^2 - \frac{19C_A}{512N_A}\lambda^{abcd}\lambda^{acbd}a \\
&+ \frac{1}{N_A}\left[\frac{31}{768}f_4^{abcd}\lambda^{apcq}\lambda^{bdpq} + \frac{9}{512}f_4^{abcd}\lambda^{apbq}\lambda^{cpdq} - \frac{25}{768}f_4^{abcd}\lambda^{acpq}\lambda^{bpdq}\right]a \\
&+ \frac{1}{4096N_A}\left[3\lambda^{abcd}\lambda^{acpq}\lambda^{bpdq} + \lambda^{abcd}\lambda^{apcq}\lambda^{bqdp}\right] + O(a^4;\lambda^4) \, . \qquad (8.95)
\end{aligned}
$$

This concludes the treatment of three loop calculations using (8.2). In the discussion section that follows we will consider, briefly, the difficulties associated with deriving the vacuum expectation values for this model using the linear composite operator method.

Chapter 9

Discussion

9.1 Complex width versus a real mass

The work presented here considered, separately, the related topics of gauge fixing and mass operators in Quantum Chromodynamics. The new calculations are carried out by making reference to the Lagrangians, (4.1) and (8.2), both of which represent an extension to the ordinary QCD Lagrangian. Each of these extensions have in common that they aim to provide a better understanding of Yang-Mills theory in the infrared sector, importantly, they both derive from the inclusion of a non-locality that is inserted into ordinary QCD.

The formalism used to develop each model is quite different. Starting with the Gribov-Zwanziger model, the extension to QCD considered here is introduced intuitively following the observation by Gribov, [20], that in order to have an analytical formulation of QCD consistent with the infrared sector, then the gauge fixing method of Faddeev and Popov must be improved. In the subsequent work by both Gribov and Zwanziger, attempts to derive an improved gauge fixing procedure rely on establishing strict criteria for the positivity of the Faddeev-Popov operator using degenerate perturbation theory. Implementing these criteria into the functional measure for non-Abelian gauge theory represents a formidable task and relies on a great deal of formal reasoning. Zwanziger does arrive at an interpretation of the gap equation which appears to be suitable for use with gauge theories. This is achieved by adding to the QCD action a term, $\gamma^2 S_1$, where

$$S_1 = -\frac{1}{2C_A}\int \mathrm{d}^d x \mathrm{d}^d y \mathrm{tr}\left[A_\mu(x)\mathcal{M}^{-1}(x,y;A)A_\nu(y)\right] . \qquad (9.1)$$

Localization using standard methods results in an action which is proven to be renormalizable by exploiting the property of BRST invariance of the massless action, that is, when $\gamma^2 = 0$. It is appropriate to comment that, the BRST invariance of the massless action did not emerge from the purely intuitive arguments that were used to derive the restriction. In order to arrive at a BRST invariant massless action, it was necessary to add a term

$$S_{(\text{BRST term})} = \int d^4x g f^{abc} \partial_\nu \bar{\omega}_\mu^{ae} (D_\nu c)^b \phi_\mu^{ec} \ , \tag{9.2}$$

on an apparently *ad hoc* basis. That is, BRST invariance is not manifestly present in Zwanziger's local implementation of a restriction to the first Gribov horizon.

In spite of these difficulties, the resulting local renormalizable action does contain all of the features of the original Gribov model fully incorporated into a QCD type Lagrangian. Also, as was seen in chapters 4 and 5, it is particularly useful for performing loop calculations to test the predictions of the model formally using renormalization group equations. Specifically, gluon suppression has been demonstrated to one loop order, and the gap equation with resulting ghost enhancement and possible implications for a confinement mechanism at two loop order are all shown to be explicitly renormalizable within the $\overline{\text{MS}}$ scheme.

Turning now to the question of a dynamically generated gluon mass, we recall that the fundamental propagators which derive from the Gribov type propagator,

$$\frac{1}{[(p^2)^2 + C_A\gamma^4]} = \frac{1}{2i\sqrt{C_A}\gamma^2}\left(\frac{1}{[p^2 - i\sqrt{C_A}\gamma^2]} - \frac{1}{[p^2 + i\sqrt{C_A}\gamma^2]}\right) \ , \tag{9.3}$$

include what is commonly referred to as a complex width. Whilst it is in no way possible to consider that these propagators include a physical mass, the Gribov parameter, γ, does have the dimension of mass. Addressing the issue of gauge fixing in the infrared sector has led to the appearance of a mass like term in the gluon propagator of QCD.

In the work that we consider here, to have a real gluon mass it was necessary to introduce a mass term into the QCD Lagrangian manually using a composite field. That is, the real mass terms considered in the second part of this thesis do not emerge from any obvious intuition about the correct way to interpret the quantization of pure Yang-Mills theory. Despite this, we obviously require that physically meaningful mass operators must be gauge invariant, as we saw in chapter 6, successfully establishing gauge invariance for such operators derives from the effectiveness of the gauge fixing procedure. Also, introducing mass terms into the QCD Lagrangian using composite operators does appear to be a natural extension when considered using the concept of dynamical symmetry breaking in asymptotically free field theories. This idea is most clearly introduced using the Gross-Neveu model, [23] a two dimensional fermion field theory with a quartic interaction that is renormalizable and asymptotically free. This model uses dynamical symmetry breaking as a mechanism for generating a fermion mass expressed in terms of the composite field $\bar{\psi}\psi$. Importantly, the resulting non-zero expectation value subsequently lowers the energy for the vacuum structure of the theory and so the presence of such a mass operator would appear to be energetically favoured. Since Quantum Chromodynamics is asymptotically free, it seems reasonable to extend the general arguments used in the Gross-Neveu model and consider the possibility of a dynamically generated gluon mass expressed in terms of a composite gluon field with general form $(A^a_\mu)^2$. In order to show that such a mechanism is likely to occur in QCD, it would be useful to show that a non-vanishing gluon condensate had an expectation value which lowered the vacuum energy for QCD.

This would require a derivation of the appropriate form of the effective action for the Yang-Mills or QCD Lagrangian, fixed in a linear covariant gauge and incorporating the gauge invariant mass operator

$$\mathcal{O} = \min_{\{u\}} \mathrm{tr} \int \mathrm{d}^4x \left(A^u_\mu\right)^2 . \tag{9.4}$$

The operator (9.4) was introduced as part of a wider review of mass operators in chapter 6, where a non-local interpretation was derived by defining the absolute minimum gauge configuration A^h_μ, (6.41). Also, due to gauge invariance of the operator, it can be expressed using a series of individually gauge invariant terms that depend on the Yang-Mills field strength tensor $F^a_{\mu\nu}$, (6.49). The first, gauge invariant, term of this operator, is that which was considered in chapters 7 and 8,

$$\mathcal{O}_1 = \frac{1}{4} \int \mathrm{d}^4x \, F^a_{\mu\nu} [(D^2)^{-1}]^{ab} F^b_{\mu\nu} . \tag{9.5}$$

In chapter 7 it was shown that the operator (9.5) has a local interpretation that may be added to the QCD Lagrangian without destroying renormalizability, albeit at the cost of adding two auxiliary fields and a quartic tensor coupling.

Expressing this operator using a local expression, not only enables exploration of renormalizability when added to the QCD Lagrangian, but allows the possibility to derive an effective action using the local composite operator method, [23]. In addition to drawing parallels between the possibility of a dynamically generated fermion or gluon mass, it is the local composite operator (LCO) formalism first used in the Gross-Neveu model that we wish to use to derive an effective action for the QCD Lagrangian incorporating a gluon mass operator. Exploiting the relative simplicity of the fermion model, we use it here to introduce the LCO method. The Lagrangian for the two-dimensional Gross-Neveu model is given by

$$L^{GN} = \bar{\psi}(\partial\!\!\!/ + J)\psi + \frac{1}{2} g^2 (\bar{\psi}\psi)^2 , \tag{9.6}$$

where $\{\psi, \bar{\psi}\}$ are fermion and anti-fermion fields, g is the coupling and J is the source term for the fields. When $J = 0$, the composite field $\bar{\psi}\psi$, vanishes along with its expectation value, and the the theory is multiplicatively renormalizable with coupling constant and wave renormalization constants. For $J \neq 0$, there are new logarithmically divergent vacuum diagrams $\propto J^2$. To have a renormalizable action incorporating the composite field $\bar{\psi}\psi$, coupled to a source J, it is necessary to introduce a new coupling constant ζ, [79],[80], such that

$$L^{GN} = \bar{\psi}(\partial\!\!\!/ + J)\psi + \frac{1}{2}g^2(\bar{\psi}\psi)^2 + \frac{1}{2}\zeta J^2 \ , \tag{9.7}$$

where all the variables in (9.7) are understood to be described using bare quantities. Renormalization is carried out according to the prescription

$$\begin{aligned} \psi_o &= Z^{1/2}\psi \\ J_o &= \frac{Z_2}{Z}J \\ g_o^2 &= \frac{Z_g}{Z^2}g^2 \\ \zeta_o J_o^2 &= \mu^{-\epsilon}(\zeta + \delta\zeta)J^2 \ , \end{aligned} \tag{9.8}$$

where Z_2 corresponds to a mass renormalization, equivalent to a wave-function renormalization of the $\bar{\psi}\psi$ operator. Now, it is possible to express the new coupling ζ, as a unique function of the quartic coupling g, where an n-loop evaluation of $\zeta(g)$ requires an $(n+1)$-loop evaluation of the renormalization group functions $\beta(g^2)$, $\lambda_2(g^2)$ and $\delta(g^2)$, where

$$\delta(g^2) = \left(\epsilon + 2\gamma_2(g^2) - \beta(g^2)\frac{\partial}{\partial g^2}\right)\delta\zeta \ . \tag{9.9}$$

Proceeding in this manner eliminates the independent coupling constant ζ and the vacuum divergences become multiplicatively renormalizable,

$$\zeta(g^2) + \delta\zeta(g^2, \epsilon) = Z_\zeta(g^2, \epsilon)\zeta(g^2) \ . \tag{9.10}$$

The associated generating functional $W[J]$ obeys a homogeneous renormalization group equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g^2)\frac{\partial}{\partial g^2} - \gamma_2(g^2)\int \mathrm{d}^2x J\frac{\delta}{\delta J}\right)W(J, g^2, \mu) = 0 \ . \tag{9.11}$$

Given this, the composite operator

$$\Delta = Z_2\bar{\psi}\psi - Z_\zeta\zeta(g^2)J \ , \tag{9.12}$$

has a finite and multiplicatively renormalizable expectation value and 2-point function,

$$\begin{aligned} \langle\Delta\rangle &= \frac{\delta W[J]}{\delta J} \\ G_\Delta(x,y) &= \frac{\delta^2 W[J]}{\delta J(x)\delta J(y)} \ . \end{aligned} \tag{9.13}$$

The new composite operator Δ differs from $\bar{\psi}\psi$ only by a c-number term, and has an identical anomalous dimension. The corresponding effective action $\Gamma(\Delta)$, is defined by

$$\Gamma[\Delta] = W[J] - \int \mathrm{d}^2x J\Delta \ , \tag{9.14}$$

which obeys the homogeneous renormalization group equation

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(g^2)+\gamma_2(g^2)\int \mathrm{d}^2x\Delta\frac{\delta}{\delta\Delta}\right)\Gamma[\Delta] \;=\; 0 \; . \tag{9.15}$$

Extending this model to a consideration of Yang-Mills/QCD which includes the non-gauge invariant composite operator, $\frac{1}{2}(A^{a\,2}_{\mu})$, in the path integral begins by defining the generating functional

$$e^{-W[J]} \;=\; \int \mathcal{D}A_\mu\mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}c\mathcal{D}\bar{c}\exp\left[\int \mathrm{d}^4x\left(L_o-\frac{1}{2}JA^{a\,2}_{o\mu}+\frac{1}{2}\zeta_oJ_o^2\right)\right] \; , \tag{9.16}$$

where L_o denotes the usual Yang-Mills/QCD Lagrangian and all quantities are bare. Two different calculations based on this approach obtain the two loop effective potential for $\langle\frac{1}{2}(A^a_\mu)^2\rangle$ in Landau gauge Yang-Mills theory, [53], and QCD, [81]. These calculations suggest that a non zero condensate is energetically favoured leading to a dynamically generated effective gluon mass. Both results are identical in the pure gauge limit of QCD, $N_f = 0$. Also, this operator has been added to the Gribov-Zwanziger Lagrangian where it was shown that it did not destroy renormalizability, [39]. Results for the operator when incorporated into the Gribov-Zwanziger Lagrangian were inconclusive and it was not possible to show that a non-vanishing value of the condensate $\langle\frac{1}{2}(A^a_\mu)^2\rangle$ reduces the vacuum energy for this model. However, it was observed that inclusion of a condensate does not affect the well known implications of a restriction to the Gribov region, that is, gluon propagator suppression and the ghost propagator enhancement.

The effective potential has also been considered in arbitrary linear covariant gauges, using a Lagrangian identical to (9.16) where propagators are derived with the value of the gauge parameter α left open, [82]. Results are consistent with those obtained in the Landau gauge, although, as expected for a non gauge invariant operator, the effective potential appears to have an explicit dependence on the gauge parameter. As such, it is only possible to regard any results obtained using the composite operator $\frac{1}{2}A^{a\,2}_\mu$ as a testing ground concerning the possibility of a physically meaningful dimension two condensate in QCD.

In order to proceed effectively, it is necessary to derive the appropriate, localized, form of the inherently non local operator $\frac{1}{2}A^2_{\text{min}}$, (9.4). As discussed, the first term in a non local expansion for this operator is given by $F^a_{\mu\nu}[(D^2)^{-1}]^{ab}F^b_{\mu\nu}$. A local interpretation of this gauge invariant operator that may be added to the QCD Lagrangian without destroying renormalizability was described in chapter 7, and in chapter 8 its anomalous dimension was computed to three loop order. Because the operator (8.1) is gauge invariant the advantages for studying this operator as an extension to the preliminary work described above for the non gauge invariant A^2 are obvious. The rather involved form of the Lagrangian, (8.2), which results from incorporating $F^a_{\mu\nu}[(D^2)^{-1}]^{ab}F^b_{\mu\nu}$ into the QCD Lagrangian in a local, renormalizable, way means that implementing the local composite operator (LCO) method is likely to be very complicated. In particular, in [80] and [53], the calculational machinery employed to express the new coupling ζ, that results from using the LCO method, as a unique function of the quartic/gauge coupling to n-loop order, required knowledge of the renormalization group functions for other variables to $(n+1)$-loop order. A study of the appropriate form of the generating functional $W(J)$ for the effective action appropriate for use with the operator (8.1) has been carried out in the recent work, [83]. An actual computation has not yet been carried out, this is primarily due to the non-availability of all the $(n+1)$-loop renormalization group functions required by the previously developed calculational machinery. If this is how the calculation of an effective action must proceed, the real stumbling block will be the two loop β-function for the quartic tensor coupling λ^{abcd}. At present, as discussed previously, we only have knowledge of this to one loop order, and our three loop calculation, described in chapter 8, is the first time it has been properly checked in a loop calculation. If in the future it is possible to overcome these difficulties then a one loop calculation for the effective action should be possible, looking further ahead, the most rigorous, theoretical, check on the result will be a two loop calculation which will require knowledge of the three loop operator anomalous dimension.

In terms of a comprehensive study for the full operator $\frac{1}{2}A^2_{\mathrm{min}}$, it is important to note that the individually gauge invariant terms in the non-local expansion (6.25) represent a leg rather than a loop expansion. That is, the more manageable first term (6.24) that we considered in chapters 7 and 8, does not furnish us with Feynman rules that are capable of describing a complete set of one particle irreducible diagrams describing the Green's functions for the operator $\frac{1}{2}A^2_{\mathrm{min}}$ to, say, one loop. However, the anomalous dimension for the complete non-local operator, using results from the localized first term described above and adding to these one loop contributions from the next term in the non-local expansion, (6.49), has been calculated in an arbitrary linear gauge to one loop in the $\overline{\mathrm{MS}}$ scheme, [62]. The result is consistent with the anomalous dimension for the non-gauge invariant operator $\frac{1}{2}A^2$ in the Landau gauge.

9.2 Recent developments concerning the Gribov-Zwanziger approach

Returning now to a discussion of the Gribov-Zwanziger approach, it is important to note that in the light of what has been discussed above, any observations made using this formalism are subject to all of the limitations mentioned for the non-gauge invariant operator $\frac{1}{2}A^2$. At the level of what we understand, a gauge fixed, infrared consistent, Yang-Mills theory considering only physically distinct gauge field configurations, that may be fully accounted for by a restriction in the path integral to the region contained inside the first Gribov horizon, is only possible in the Landau gauge. This limitation arose from the requirement that the Faddeev-Popov (FP) operator is required to be Hermitian in the process of identification and implementation of a meaningful Gribov region, both for the Gribov and the subsequent Zwanziger treatments. Only in the Landau gauge does the FP operator display this property.

Any analytical approach to the low energy behaviour in Yang-Mills theory must pay close attention to results obtained using inherently non-perturbative techniques such as, Dyson-Schwinger equations (DSE), and numerical lattice studies. Lattice simulations of gauge dependent quantities are known to suffer from the problem of Gribov copies, particularly in the infrared regime, although it is generally believed that the effects are quantitative rather than qualitative, [86]. Whilst the effects of the Gribov ambiguity on the gluon propagator are believed to be contained within the statistical error for the simulation, it is possible that implications for the ghost propagator can become pronounced in the infrared. Until recently there has been good agreement between data from numerical lattice studies and analytical results arising from the Gribov-Zwanziger action in the Landau gauge. That is

- an infrared suppressed gluon propagator vanishing at zero momentum
- an infrared enhanced ghost propagator.

From here onwards this will be referred to as the scaling solution. However, recent numerical studies, obtained at large volumes, obtain results different from that deriving from the Gribov-Zwanziger action, [84],[85],[86],[87],[88],[89]. In other words

- an infrared suppressed gluon propagator not vanishing at zero momentum
- a ghost propagator of fundamental type in the infrared in agreement with the ultraviolet ghost propagator seen in perturbative QCD

From here onwards this will be referred to as the decoupling solution. Despite the fact that, until now, all relevant numerical studies had agreed with the established scaling solution, a subsequent study using DSE methods produced results for the gluon and ghost propagators in qualitative agreement with the recent lattice data, [86]. Assuming that recent lattice results do reach into the far infrared region and that Gribov copies are under control, it is believed that the results can be reproduced analytically by introducing an additional dynamical effect into the Gribov-Zwanziger action in what has come to be known as the *decoupling* solution, [90],[91].

The fields $\phi \in \{\varphi_\mu^{ab}, \bar{\varphi}_\mu^{ab}, \omega_\mu^{ab}, \bar{\omega}_\mu^{ab}\}$ are necessary to realize a local interpretation of the horizon function

$$S_{\text{horizon}} = \gamma^4 g^2 \int \mathrm{d}^4x f^{abc} A_\mu^b \left[(-\partial_\nu D_\nu)^{-1}\right]^{ad} f^{dec} A_\mu^e \ . \tag{9.17}$$

In order to account fully for the non-local dynamics dictated by (9.17) it seems reasonable that, at the quantum level, there might be non-trivial dynamics associated with the additional localizing fields. When one considers also that the Bosonic ϕ_μ^{ab} are coupled directly to the gauge field in the tree level action, we realize that dynamical effects in the ϕ_μ^{ab} sector may have dynamical implications for the gluon sector, which can in turn modify the behaviour of the Faddeev-Popov ghosts. This situation may be explored by considering the possibility of a dynamical mass generation for the ϕ_μ^{ab} fields. By introducing a local composite operator into the Gribov-Zwanziger action, [90],[91],

$$\begin{aligned} S &= S_{GZ} + S_{\phi\bar{\phi}} \\ S_{\phi\bar{\phi}} &= \int d^4x \left[s(-J\bar{\omega}_\mu^{ab}\phi_\mu^{ab}) + \rho\frac{J^2}{2} \right] \\ &= \int d^4x \left[-J(\bar{\phi}_\mu^{ab}\phi_\mu^{ab} - \bar{\omega}_\mu^{ab}\omega_\mu^{ab}) + \rho\frac{J^2}{2} \right] . \end{aligned} \tag{9.18}$$

The BRST invariant operator

$$(\bar{\phi}_\mu^{ab}\phi_\mu^{ab} - \bar{\omega}_\mu^{ab}\omega_\mu^{ab})$$

fits quite naturally into the theory, the resulting action is renormalizable to all orders and does not generate any new types of divergence. Having, retrospectively, introduced an explicit mass term into the Gribov-Zwanziger Lagrangian, it should be expected that the additional term will affect the general form of the associated propagators. For the modified action, the gluon propagator is derived by referring to the quadratic piece $\mathcal{L}_{quad}^{GZ}$, where the authors chose to introduce an explicit mass M for the fields, $\{\phi_\mu^{ab}, \bar{\phi}_\mu^{ab}\}$, in the presence of the local composite operator $\frac{1}{2}A^2$ with corresponding gluon mass m. In a preliminary study, the structure taken by the gluon propagator is examined by removing the auxiliary fields using their equations of motion in momentum space, such that

$$\frac{\partial \mathcal{L}_{quad}^{GZ}}{\partial \phi_\mu^{ab}} = 0 \quad , \quad \frac{\partial \mathcal{L}_{quad}^{GZ}}{\partial \bar{\phi}_\mu^{ab}} = 0 \ , \tag{9.19}$$

leading to

$$\phi_\mu^{bc} = \bar{\phi}_\mu^{bc} = \frac{1}{[p^2 - M^2]}\gamma^2 g f^{abc} A_\mu^a \ , \tag{9.20}$$

where M represents the new mass term. Substituting these values for the fields, $\{\phi_\mu^{ab}, \bar{\phi}_\mu^{ab}\}$, back into $\mathcal{L}_{quad}^{GZ}$, inverting the resulting matrix and taking the Landau gauge limit, $\alpha \to 0$, the modified action results in a modified tree level gluon propagator where we choose to set the gluon mass equal to zero, $m = 0$,

$$\langle A_\mu^a(p) A_\nu^b(-p) \rangle = \frac{p^2 + M^2}{[(p^2)^2 + M^2 p^2 + C_A \gamma^4]} \left[\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right] \delta^{ab}. \tag{9.21}$$

From (9.21), we can immediately see that the tree level gluon propagator of the decoupling solution to the Gribov-Zwanziger Lagrangian enjoys the property of infrared suppression and is non vanishing at the origin, consistent with the data collected in recent lattice simulations.

A one loop study of the Faddeev-Popov ghost propagator, again incorporating a retrospective mass term, M for the fields $\{\phi^{ab}_\mu, \bar{\phi}^{ab}_\mu\}$ is carried out by implementing the no pole condition, following the method of chapter 2, [20],[24]. Recalling that for the ghost propagator we have, in terms of the no pole condition

$$\mathcal{G}(k;A) \approx \frac{1}{k^2}\frac{1}{1-\sigma(k,A)} , \tag{9.22}$$

where, again in the Landau gauge, and this time including the modified gluon propagator (9.21)

$$\begin{aligned} \sigma(k,A) &= \frac{N}{N^2-1}\frac{1}{k^2}\int\frac{d^4p}{(2\pi)^4}\frac{(k-p)_\mu k_\nu}{(k-p)^2}\langle A^a_\mu(-p)A^a_\nu(p)\rangle \\ &= \frac{Nk_\mu k_\nu}{k^2}\int\frac{d^4p}{(2\pi)^4}\frac{1}{(k-p)^2}\frac{p^2+M^2}{[(p^2)^2+M^2p^2+C_A\gamma^4]}\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right) . \end{aligned} \tag{9.23}$$

Repeating the original analysis results in a one loop correction to the ghost propagator where, for $M^2 \neq 0$, the ghost propagator is no longer enhanced but behaves like $1/k^2$, also consistent with the decoupling solution.

From the perspective of confinement, in particular, the Kugo-Ojima criterion, the decoupling solution is not consistent with this scenario. The modified Gribov-Zwanziger model now represents an improved Landau gauge fixing procedure, it may be used to derive propagators which, in the infrared, have a behaviour consistent with that observed in recent lattice and DSE studies. The modified action appears to say less about confinement. Given this, it should also be noted that there are difficulties associated with lattice and DSE methods in the far infrared region. In particular, for lattice studies it is believed that the continuum limit is not fully under control and that DSE studies are delicate to boundary conditions. That is, for many, the respective merits or validity of a decoupling versus a scaling solution remains an open question. If it is established that the decoupling solution is in fact the correct picture, then deeper investigations are likely to require a consideration of Green's functions described using propagators with a complex width and a real mass. As such, we believe that our investigations incorporating massive quarks into the two loop gap equation using a complex dilogarithm solution represent first steps in performing this type of calculation.

Appendix A

Long Gribov derivations

In this short appendix we include the full detail for the explicit identification of a Gribov copy, and the derivation of the no pole condition for the Faddeev-Popov ghost propagator.

A.1 Generic Gribov copy

For the eigenvalue problem with potential A_μ identified in chapter 2, the standard perturbation theory of quantum mechanics is applied to the expression

$$-\partial_\mu(\partial_\mu\psi + [C_\mu + a_\mu, \psi]) = -(\partial^2\psi + [C_\mu, \partial_\mu\psi] + \partial_\mu[a_\mu, \psi]) = \epsilon(C)\psi + \epsilon(a)\psi \quad . \quad (A.1)$$

We already know that there exists a solution, ϕ_0, such that $\epsilon(C)\phi_0 = 0$. Within perturbation theory, consideration of the eigenvalues for modes of the Faddeev-Popov operator with potential A_μ can be reduced to a consideration of the eigenvalue of the operator $\partial_\mu[a_\mu, \cdot]$, for the zero mode of the Faddeev-Popov operator with potential C_μ. That is,

$$\epsilon(a) = -\frac{\langle\phi_0|\partial_\mu[a_\mu, \phi_0]\rangle}{\langle\phi_0|\phi_0\rangle} = -\frac{\mathrm{tr}\int \mathrm{d}^4x(\phi_0\partial_\mu[a_\mu, \phi_0])}{\mathrm{tr}\int \mathrm{d}^4x(\phi_0\phi_0)} \quad . \quad (A.2)$$

The field $A_\mu = C_\mu + a_\mu$ is transformed according to

$$A_\mu \to \tilde{A}_\mu = u^\dagger\partial_\mu u + u^\dagger A_\mu u = A_\mu + u^\dagger(\partial_\mu u + [A_\mu, u]) \quad , \quad (A.3)$$

taking u close to unity. A first order infinitesimal gauge transformation would give

$$\tilde{A}_\mu = A_\mu + D_\mu(A)\sigma \quad . \quad (A.4)$$

In order for the transformed fields to satisfy an identical gauge condition we would require that

$$\partial_\mu D_\mu(A)\sigma = -\mathcal{M}(A)\sigma = 0 \quad , \quad (A.5)$$

which has no solutions for non-zero σ because the field A_μ is not located on the Gribov horizon. We must retain terms of u up to second order

$$u = 1 + \sigma + \frac{1}{2}\sigma^2 + O(\sigma^3) \quad , \quad u^\dagger = 1 - \sigma + \frac{1}{2}\sigma^2 + O(\sigma^3) \quad , \tag{A.6}$$

and after a little algebra we arrive at

$$A'_\mu = A_\mu + D_\mu(A)\sigma - \frac{1}{2}[\sigma, \partial_\mu\sigma + [A_\mu, \sigma]] + O(\sigma^3) \quad , \tag{A.7}$$

where $D_\mu(A)\sigma = \partial_\mu\sigma + [A_\mu, \sigma]$. From (A.7) we see that the condition for the existence of a Gribov copy; a pair of gauge transformed fields that satisfy an identical gauge constraint, is given by

$$\partial_\mu D_\mu(A)\sigma - \frac{1}{2}\partial_\mu[\sigma, \partial_\mu\sigma + [A_\mu, \sigma]] = 0 \quad , \tag{A.8}$$

or, more definitely, in terms of the transverse field located on the horizon, C_μ, and associated zero mode of $\mathcal{M}(C)$, ϕ_0,

$$D_\mu(C)\phi_0 = D_\mu(A)\sigma - \frac{1}{2}[\sigma, \partial_\mu\sigma + [A_\mu, \sigma]] \quad . \tag{A.9}$$

Taking the divergence of both sides of (A.9), we obtain the condition to be fulfilled in order that A_μ and $\tilde{A}_\mu$ satisfy the same divergence condition $\partial_\mu A_\mu = \partial_\mu \tilde{A}_\mu$. That is,

$$\partial_\mu D_\mu(A)\sigma = \frac{1}{2}\partial_\mu[\sigma, \partial_\mu\sigma + [A_\mu, \sigma]] \quad . \tag{A.10}$$

By setting

$$\begin{aligned} \sigma &= \phi_0 + \bar{\sigma} \\ A_\mu &= C_\mu + a_\mu \quad , \end{aligned} \tag{A.11}$$

where, $\bar{\sigma}$ and a_μ are respectively small with respect to ϕ_0 and A_μ, it is possible to analyze (A.10) iteratively. It is useful to introduce the expansion parameter λ,

$$\begin{aligned} \sigma &= \lambda\phi_0 + \lambda^2\bar{\sigma} \\ A_\mu &= C_\mu + a_\mu \quad , \end{aligned} \tag{A.12}$$

which is set equal to one at the end. Inserting this parametrization into (A.10) gives

$$\begin{aligned} & \partial^2\left(\lambda\phi_0 + \lambda^2\bar{\sigma}\right) + \partial_\mu\left[\, C_\mu + \lambda a_\mu, \lambda\phi_0 + \lambda^2\bar{\sigma}\right] \\ = \; & \frac{1}{2}\partial_\mu\left[\lambda\phi_0 + \lambda^2\bar{\sigma}\, , \partial_\mu\left(\lambda\phi_0 + \lambda^2\bar{\sigma}\right)\right] \\ & + \frac{1}{2}\partial_\mu\left[\lambda\phi_0 + \lambda^2\bar{\sigma}\, , \left[C_\mu + \lambda a_\mu, \lambda\phi_0 + \lambda^2\bar{\sigma}\right]\right] \end{aligned} \tag{A.13}$$

so that, up to terms of the order λ^4

$$\begin{aligned} & \lambda^2\partial_\mu\left(\partial_\mu\bar{\sigma} + [\, C_\mu, \bar{\sigma}]\right) + \lambda^2\partial_\mu\,[\, a_\mu, \phi_0] + \lambda^3\partial_\mu\,[\, a_\mu, \bar{\sigma}] \\ = \; & \frac{\lambda^2}{2}\partial_\mu\,[\phi_0\, , \partial_\mu\phi_0 + [C_\mu, \phi_0]] + \frac{\lambda^3}{2}\partial_\mu\,[\phi_0, \partial_\mu\bar{\sigma} + [C_\mu, \bar{\sigma}]] \\ & + \frac{\lambda^3}{2}\partial_\mu\,[\bar{\sigma}, \partial_\mu\phi_0 + [C_\mu, \phi_0]] + \frac{\lambda^3}{2}\partial_\mu\,[\phi_0, [a_\mu, \phi_0]] + O(\lambda^4) \quad . \end{aligned} \tag{A.14}$$

In particular, to the first order λ^2, we find

$$\partial_\mu D_\mu(C)\tilde{\sigma} + \partial_\mu [\, a_\mu, \phi_0] = \frac{1}{2}\partial_\mu [\phi_0, D_\mu(C)\phi_0] \ , \tag{A.15}$$

from which it follows that

$$\mathrm{tr}\int \mathrm{d}^4x \,(\phi_0 \partial_\mu D_\mu(C)\tilde{\sigma} + \phi_0\partial_\mu [\, a_\mu, \phi_0]) = \frac{1}{2}\mathrm{tr}\int \mathrm{d}^4x\,(\phi_0\partial_\mu [\phi_0, D_\mu(C)\phi_0]) \ . \tag{A.16}$$

Moreover, due to property $\partial_\mu D_\mu(C) = D_\mu(C)\partial_\mu$, we have

$$\mathrm{tr}\int \mathrm{d}^4x \phi_0\partial_\mu D_\mu(C)\tilde{\sigma} = \mathrm{tr}\int \mathrm{d}^4x\,(\partial_\mu D_\mu(C)\phi_0)\,\tilde{\sigma} = 0 \ . \tag{A.17}$$

As a consequence, condition (A.16) reads

$$\mathrm{tr}\int \mathrm{d}^4x\,(\phi_0\partial_\mu [\, a_\mu, \phi_0]) = \frac{1}{2}\mathrm{tr}\int d^4x\,(\phi_0\partial_\mu [\phi_0, D_\mu(C)\phi_0]) \ . \tag{A.18}$$

It remains now to check on which side of the horizon l_1 the equivalent field $\tilde{A}_\mu$ lies. Let us rewrite $\tilde{A}_\mu$ as

$$\tilde{A}_\mu = A_\mu + D_\mu(C)\phi_0 \ = C_\mu + a_\mu + D_\mu(C)\phi_0 = C_\mu + \tilde{a}_\mu \ . \tag{A.19}$$

As done before, we evaluate the shift $\epsilon(\tilde{a})$ of the eigenvalue of the Faddeev-Popov operator from zero. Treating the field

$$\tilde{a}_\mu = a_\mu + D_\mu(C)\phi_0 \ , \tag{A.20}$$

as a perturbation, one obtains

$$\epsilon(\tilde{a}) = -\frac{\left\langle \phi_0 \mid \partial_\mu \left[a'_\mu, \phi_0\right]\right\rangle}{\langle \phi_0 \mid \phi_0\rangle} = -\frac{\mathrm{tr}\int \mathrm{d}^4x\,(\phi_0\partial_\mu [a_\mu, \phi_0] + \phi_0\partial_\mu [D_\mu(C)\phi_0, \phi_0])}{\mathrm{tr}\int \mathrm{d}^4x\,(\phi_0\phi_0)} \ . \tag{A.21}$$

Furthermore, from (A.18), it follows

$$\epsilon(\tilde{a}) = \frac{\mathrm{tr}\int d^4x\,(\phi_0\partial_\mu [a_\mu, \phi_0])}{\mathrm{tr}\int d^4x\,(\phi_0\phi_0)} = -\epsilon(a) \ . \tag{A.22}$$

Thus, if A_μ, close to l_1, is located in C_0, $\epsilon(a) > 0$, there is an equivalent field, $\tilde{A}_\mu = A_\mu + D_\mu(C)\phi_0$, close to l_1, which is located in C_1, $\epsilon(\tilde{a}) = -\epsilon(a) < 0$. Also, it is worth mentioning that this derivation can be generalized to fields close to any horizon l_n. This concludes the proof of the statement.

A.2 The no pole condition

We proceed with a characterization of the factor $\mathcal{V}(C_0)$ by denoting $\mathcal{G}(k;A)$ by the colour singlet Fourier transform of $[-\partial_\mu(\partial_\mu - f^{abc}A^a_\mu)]^{-1}$,

$$\mathcal{G}(k;A) = \sum_{ab}\frac{\delta^{ab}}{N_A^2-1}\left\langle k\left|\left[-\partial_\mu\left(\partial_\mu\delta^{ab} - f^{abc}A^c_\mu\right)\right]^{-1}\right|k\right\rangle \ , \tag{A.23}$$

and will require that $\mathcal{G}(k;A)$ has no poles for non vanishing momenta k. The expression for the connected, colour singlet, ghost two-point function is given by

$$\begin{aligned}\sum_{ab}\frac{\delta^{ab}\langle\bar{c}^a(x)c^a(y)\rangle}{N_A^2-1} &= \mathcal{N}\int\mathcal{D}A_\mu\mathcal{D}\bar{c}\mathcal{D}c\,\delta(\partial A)\frac{\bar{c}^a(x)c^a(y)}{N_A^2-1}e^{-(S_{YM}+\int \mathrm{d}^4x\bar{c}^a\partial_\mu D_\mu^{ab}c^b)} \\ &= \mathcal{N}\int\mathcal{D}A_\mu\delta(\partial A)e^{-S_{YM}}\mathcal{G}(x,y;A)\,, \qquad \text{(A.24)}\end{aligned}$$

where the gauge field A_μ^a is a classical external field. Using Wick's theorem it is possible to evaluate $\langle\bar{c}^a(x)c^b(y)\rangle$ to second order in perturbation theory,

$$\begin{aligned}\langle\bar{c}^a(x)c^b(y)\rangle &= \langle\bar{c}_0^a(x)c_0^b(y) \\ &\times\Big[1+\int \mathrm{d}^4x_1(\partial_{x_1}^\mu\bar{c}_0^m(x_1))f^{mnp}A_\mu^n(x_1)c_0^p(x_2) \\ &+\frac{1}{2}\int \mathrm{d}^4x_1\mathrm{d}^4x_2 \\ &\times(\partial_{x_1}^\mu\bar{c}_0^m(x_1))f^{mnp}A_\mu^n(x_1)c_0^p(x_1)(\partial_{x_2}^\nu\bar{c}_0^q(x_2))f^{qrt}A_\nu^r(x_2)c_0^t(x_2)\Big]\rangle\,, \\ &\qquad \text{(A.25)}\end{aligned}$$

where $\bar{c}_0^a(x)$ and $c_0^b(y)$ represent free ghost fields. Wick contraction reduces this expression to products of free field ghost two-point functions,

$$\begin{aligned}\langle\bar{c}^a(x)c^b(y)\rangle &= \langle\bar{c}_0^a(x)c_0^b(y)\rangle-\int \mathrm{d}^4x_1f^{mnp}A_\mu^n(x_1)\langle\bar{c}_0^a(x)c_0^m(x_1)\rangle\partial_{x_1}^\mu\langle\bar{c}_0^p(x_1)c_0^b(y)\rangle \\ &+\int \mathrm{d}^4x_1\mathrm{d}x_2f^{mnp}A_\mu^n(x_1)f^{qrt}A_\nu^r(x_2) \\ &\times\langle\bar{c}_0^a(x)c_0^p(x_1)\rangle\partial_{x_1}^\mu\langle\bar{c}_0^m(x_1)c_0^t(x_2)\rangle\partial_{x_2}^\nu\langle\bar{c}_0^q(x_2)c_0^b(y)\rangle\,. \\ &\qquad \text{(A.26)}\end{aligned}$$

Denoting the free field ghost two-point function in terms of the free ghost propagator $\langle\bar{c}_0^a(x)c_0^b(y)\rangle=\delta^{ab}\mathcal{G}_0(x-y)$,

$$\begin{aligned}\langle\bar{c}^a(x)c^b(y)\rangle &= \delta^{ab}\mathcal{G}_0(x-y)-\int \mathrm{d}^4x_1\mathcal{G}_0(x-x_1)\partial_{x_1}^\mu\mathcal{G}_0(x_1-y)f^{bna}A_\mu^n(x_1) \\ &+\int \mathrm{d}^4x_1\mathrm{d}^4x_2 \\ &\times\mathcal{G}_0(x-x_1)\partial_{x_1}^\mu\mathcal{G}_0(x_1-x_2)\partial_{x_2}^\nu\mathcal{G}_0(x_2-y)f^{tna}A_\mu^n(x_1)f^{brt}A_\nu^r(x_2)\,. \\ &\qquad \text{(A.27)}\end{aligned}$$

Finally, for $\mathcal{G}(x,y;A)$ we obtain

$$\begin{aligned}\mathcal{G}(x,y;A) &= \mathcal{G}_0(x-y)-\frac{1}{N_A^2-1}\int \mathrm{d}^4x_1\mathcal{G}_0(x-x_1)\partial_{x_1}^\mu\mathcal{G}_0(x_1-y)f^{ana}A_\mu^n(x_1) \\ &+\frac{1}{N_A^2-1} \\ &\times\int \mathrm{d}^4x_1\mathrm{d}^4x_2\,\big(\mathcal{G}_0(x-x_1)\partial_{x_1}^\mu\mathcal{G}_0(x_1-x_2) \\ &\quad\times\partial_{x_2}^\nu\mathcal{G}_0(x_2-y)f^{tna}A_\mu^n(x_1)f^{art}A_\nu^r(x_2)\big)\,, \\ &\qquad \text{(A.28)}\end{aligned}$$

because of the antisymmetry of the structure constants

$$\begin{aligned} \mathcal{G}(x,y;A) &= \mathcal{G}_0(x-y) - \frac{N_A}{N_A^2-1} \\ &\times \int \mathrm{d}^4x_1 \mathrm{d}^4x_2 \mathcal{G}_0(x-x_1)\partial^\mu_{x_1}\mathcal{G}_0(x_1-x_2)\partial^\nu_{x_2}\mathcal{G}_0(x_2-y)A^a_\mu(x_1)A^a_\nu(x_2) \ . \end{aligned} \tag{A.29}$$

It now remains to take the Fourier transform of (A.29) in a finite volume V given by a four dimensional hypercube, taking the thermodynamic limit, $V \to \infty$, at the end. The Fourier transform of the fields $\phi \in \{A^a_\mu, c^a, \bar{c}^a\}$ is given by

$$\phi^a(x) = \frac{1}{\sqrt{V}} \sum_q \phi^a(q) \exp(-iqx) \ , \tag{A.30}$$

in the thermodynamic limit

$$\sum_q \to V \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \ . \tag{A.31}$$

The Fourier transform of the free ghost propagator is given by

$$\begin{aligned} \langle \bar{c}^a_0(x) c^b_0(y) \rangle &= \mathcal{N} \int \mathcal{D}c\mathcal{D}\bar{c}\bar{c}^a_0(x)c^b_0(y) \exp\left(-\int \mathrm{d}^4x \bar{c}^a \partial^2 c^a\right) \\ &= \frac{1}{V}\sum_{p,q} e^{i(qx+py)} \mathcal{N} \int \mathcal{D}c\mathcal{D}\bar{c}\bar{c}^a_0(q)c^b_0(p) \exp\left(-\sum_k \bar{c}^a(k)k^2c^a(-k)\right) \\ &= \frac{1}{V}\sum_q e^{iq(x-y)} \frac{\delta^{ab}}{q^2} \ . \end{aligned} \tag{A.32}$$

This gives for $\mathcal{G}_0(x-y)$,

$$\mathcal{G}_0(x-y) = \frac{1}{V}\sum_q e^{i(x-y)} \frac{1}{q^2} \tag{A.33}$$

We are now ready to evaluate the Fourier transform of $\mathcal{G}_0(x,y;A)$. Using

$$\mathcal{G}(k;A) = \frac{1}{V}\int \mathrm{d}^4x \mathrm{d}^4y \mathcal{G}(x,y;A) \ , \tag{A.34}$$

gives,

$$\begin{aligned} \mathcal{G}(k;A) &= \frac{1}{V}\int \mathrm{d}^4x \mathrm{d}^4y e^{ik(x-y)}\mathcal{G}_0(x-y) \\ &-\frac{N_A}{N_A^2-1}\frac{1}{V}\int \mathrm{d}^4x \mathrm{d}^4y \mathrm{d}^4x_1 \mathrm{d}^4x_2 e^{ik(x-y)} \\ &\times \left[\mathcal{G}_0(x-x_1)\partial^\mu_{x_1}\mathcal{G}_0(x_1-x_2)\partial^\nu_{x_2}\mathcal{G}_0(x_2-y)A^a_\mu(x_1)A^a_\nu(x_2)\right] \ . \end{aligned} \tag{A.35}$$

Thus

$$\mathcal{G}(k;A) = \frac{1}{V^2}\sum_q \int \mathrm{d}^4x \mathrm{d}^4y e^{ix(k+q)} e^{-iy(k+q)} \frac{1}{q^2}$$

$$
\begin{aligned}
& +\frac{N_A}{N_A^2-1}\frac{1}{V^5}\sum_{q,p,l,u,r}\int \mathrm{d}^4x\mathrm{d}^4y\mathrm{d}^4x_1\mathrm{d}^4x_2 \\
& \times\left[e^{ix(k+q)}e^{-iy(k+l)}e^{ix_1(p-q+u)}e^{ix_2(l-p+r)}\frac{1}{q^2}\frac{p_\mu}{p^2}\frac{l_\nu}{l^2}A_\mu^a(u)A_\nu^a(r)\right] \\
= & \frac{1}{V^2}\sum_q\int \mathrm{d}^4xe^{ix(k+q)}\frac{1}{q^2}\delta_{(q+k)0} \\
& +\frac{N_A}{N_A^2-1}\frac{1}{V^5}\sum_{q,p,l,u,r}\left[\delta_{q-k}\delta_{l-k}\delta_{(p-q+u)0}\delta_{(l-p+r)0}\frac{1}{q^2}\frac{p_\mu}{p^2}\frac{l_\nu}{l^2}A_\mu^a(u)A_\nu^a(r)\right] \\
= & \frac{1}{k^2}+\frac{N_A}{N_A^2-1}\frac{1}{V}\frac{1}{k^2}\frac{-k_\nu}{k^2}\sum_p\frac{p_\mu}{p^2}A_\mu^a(-p-k)A_\nu^a(p+k) \\
= & \frac{1}{k^2}+\frac{N_A}{N_A^2-1}\frac{1}{V}\frac{1}{k^2}\frac{-k_\nu}{k^2}\sum_p\frac{(p-k)_\mu}{(p-k)^2}A_\mu^a(-p)A_\nu^a(p) \\
= & \frac{1}{k^2}\left(1+\frac{N_A}{N_A^2-1}\frac{1}{V}\frac{1}{k^2}\sum_p\frac{(k-p)_\mu k_\nu}{(p-k)^2}A_\mu^a(-p)A_\nu^a(p)\right) \\
= & \frac{1}{k^2}\left(1+\sigma(k,A)\right)\approx\frac{1}{k^2}\frac{1}{(1-\sigma(k,A))} \quad . \qquad \text{(A.36)}
\end{aligned}
$$

$$
\mathcal{G}(k;A)\approx\frac{1}{k^2}\frac{1}{(1-\sigma(k,A))} \quad . \tag{A.37}
$$

$$
\sigma(k,A)=\frac{N_A}{N_A^2-1}\frac{1}{V}\frac{1}{k^2}\sum_p\frac{(k-p)_\mu k_\nu}{(p-k)^2}A_\mu^a(-p)A_\nu^a(p) \tag{A.38}
$$

In the thermodynamic limit, $V\to\infty$,

$$
\sigma(k,A)=\frac{N_A}{N_A^2-1}\frac{1}{k^2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{(k-q)_\mu k_\nu}{(q-k)^2}A_\mu^a(-q)A_\nu^a(q) \quad , \tag{A.39}
$$

and this concludes the derivation of $\sigma(k,A)$.

Bibliography

[1] C.N. Yang & R.L. Mills, Phys. Rev. **96** (1954), 191.

[2] L.D. Faddeev & V.N. Popov Phys. Lett. **B25** (1967), 29.

[3] M. Veltman, Nucl. Phys. **B21** (1970), 288.

[4] G. 't Hooft, Nucl. Phys. **B33** (1971), 173.

[5] G. 't Hooft, Nucl. Phys. **B35** (1971), 167.

[6] R.P. Feynman, Acta. Phys. Pol. **24** (1963), 262.

[7] S. Weinberg, Phys. Rev. Lett. **19** (1967), 1264.

[8] A. Salam, Elementary Particle Theory, Ed. N. Svarathon, (Almquist & Forlag, Stockholm, 1968).

[9] M. Gell-Mann, Phys. Lett. **8** (1964), 214.

[10] C.G. Callan, Phys. Rev. **D2** (1970), 1541.

[11] K. Symanzik, Comm. Math. Phys. **18** (1970), 227.

[12] D.J. Gross & F. Wilczek, Phys. Rev. Lett. **30** (1973), 1343.

[13] H.D. Politzer, Phys. Rev. Lett. **30** (1973), 1346.

[14] F.J. Dyson, Phys. Rev. **75** (1949), 1736.

[15] J. Schwinger, Proc. Nat. Acad. Sc. **37** (1951), 452 & 455.

[16] K.G. Wilson, Phys. Rev. **D10** (1974), 2445.

[17] C. Becchi, A. Rouet & R. Stora, Ann. Phys. **98** (1976), 287.

[18] I.V. Tyutin, Lebedev-75-39; arXiv:0812.0580, [hep-th].

[19] T. Kugo & I. Ojima, Prog. Theor. Phys. Suppl. **66** (1979).

[20] V.N. Gribov, Nucl. Phys. **B139** (1978), 1.

[21] D. Zwanziger, Nucl. Phys. **B399** (1993), 477.

[22] Y. Nambu & G. Jona-Lasino, Phys. Rev. **122** (1961), 345.

[23] D.J. Gross & A. Neveu, Phys. Rev. **D10** (1974), 3235.

[24] R.F. Sobreiro & S.P. Sorella, hep-th/0504095.

[25] F.S. Henyey, Phys. Rev. **D20** (1979), 1460.

[26] P. van Baal, Nucl. Phys. **B369** (1992), 259.

[27] D. Zwanziger, Nucl. Phys. **B209** (1982), 336.

[28] I.M. Singer, Comm. Math. Phys. **60** (1978), 7.

[29] D. Zwanziger, Nucl. Phys. **B321** (1989), 591.

[30] D. Zwanziger, Nucl. Phys. **B323** (1989), 513.

[31] D. Zwanziger, Nucl. Phys. **B378** (1992), 525.

[32] J. Zinn-Justin, Quantum field theory and critical phenomena (Oxford, 1989).

[33] C. Itzykson & J.-B. Zuber, Quantum field theory (McGraw-Hill, New York, 1980).

[34] N. Maggiore & M. Schaden, Phys. Rev. **D50** (1994), 6616.

[35] A. Blasi, O. Piguet & S.P. Sorella, Nucl. Phys. **B356** (1991), 154.

[36] Y.M.P Lam, Phys. Rev. **D6** (1972), 2145.

[37] Y.M.P Lam, Phys. Rev. **D6** (1972), 2161.

[38] T.E. Clark & J.H. Lowenstein, Nucl. Phys. **B113** (1976), 109.

[39] D. Dudal, R.F. Sobreiro, S.P. Sorella & H. Verschelde, Phys. Rev. **D72** (2005), 014016.

[40] J.A. Gracey, Phys. Lett. **B632** (2006), 282.

[41] J.A. Gracey, JHEP **05** (2006), 052.

[42] L.H. Ryder, Quantum Field Theory (Cambridge, 1985).

[43] P. Nogueira, J. Comput. Phys. **105** (1993), 279.

[44] J. Vermaseren, math-ph/0010025

[45] S.A. Larin & J.A.M Vermaseren, Phys. Lett. **B303** (1993), 344.

[46] C. Ford, I. Jack & D.R.T. Jones, Nucl. Phys. **B378** (1992), 373.

[47] A.I. Davydychev & J.B. Tausk, Nucl. Phys. **B397** (1993), 123.

[48] D.J. Broadhurst, Z. Phys. **C47** (1990), 115.

[49] A.I. Davydychev & R. Delbourgo, J. Math. Phys. **39** (1998), 4299.

[50] D. Choudhury, R. Gandhi, J.A. Gracey & B. Mukhopadhyaya, Phys. Rev. **D50** (1994), 3468.

[51] L. Lewin, Dilogarithms and associated functions (Macdonald, London, 1958).

[52] J.M. Cornwall, Phys. Rev. **D26** (1982), 1453.

[53] H. Verschelde, K. Knecht, K. Van Acoleyen & M. Vanderkelen, Phys. Lett. **B516** (2001), 307.

[54] M.J. Lavelle & M. Schaden, Phys. Lett. **B208** (1988), 297.

[55] F.V. Gubarev & V.I. Zakharov, Phys. Lett. **B501** (2001), 28.

[56] F.V. Gubarev, L. Stodolsky & V.I. Zakharov, Phys. Rev. Lett. **86** (2001), 2220.

[57] H. Ruegg & M. Ruiz-Altaba, Int. J. Mod. Phys. **A19** (2004), 3265.

[58] M.A.L. Capri, D. Dudal, J.A. Gracey, V.E.R. Lemes, R.F. Sobreiro, S.P. Sorella & H. Verschelde, Phys. Rev. **D72** (2005), 105016.

[59] M.A.L. Capri, D. Dudal, J.A. Gracey, V.E.R. Lemes, R.F. Sobreiro, S.P. Sorella & H. Verschelde, Phys. Rev. **D74** (2006), 045008.

[60] D. Zwanziger, Nucl. Phys. **B345** (1990), 461.

[61] G. Dell'Antonio & D. Zwanziger, Comm. Math. Phys. **138** (1991), 291.

[62] J.A. Gracey, Phys. Lett. **B651** (2007) 253.

[63] M. Lavelle and D. McMullan, Phys. Rep. **279** (1997), 1.

[64] O. Piguet & S.P. Sorella, Lect. Notes Phys. **M28** (1995), 1.

[65] N. Nakanishi & I. Ojima, Z. Phys. **C6** (1980), 155.

[66] C. Lucchesi, O. Piguet, & K. Sibold, Int. J. Mod. Phys. **A2** (1987), 385.

[67] S.A. Larin, F.V. Tkachov & J.A.M. Vermaseren, NIKHEF-H-91-18 (1991).

[68] F.V. Tkachov, Phys. Lett. **B100** (1981), 1.

[69] K.G. Chetyrkin & F.V. Tkachov. Nucl. Phys. **192B** (1981), 159.

[70] S.A. Larin, NIKHEF-00-032 (2000).

[71] S.A. Larin & J.A.M. Vermaseren, Phys. Lett. **B303** (1993), 334.

[72] D.R.T. Jones, Nucl. Phys. **B75** (1974), 531.

[73] W.E. Caswell, Phys. Rev. Lett. **33** (1974), 244.

[74] T. van Ritbergen, A.N. Schellekens & J.A.M. Vermaseren, Int. J. Phys. **A14** (1999), 41.

[75] O.V. Tarasov & A.A. Vladimirov, Sov. J. Nucl. Phys. **25** (1997), 585.

[76] E. Egorian & O.V. Tarasov, Theor. Math. Phys. **41** (1979), 863.

[77] O.V. Tarasov, A.A. Vladimirov & A.Yu. Zharkov, Phys. Lett. **B93** (1980), 429.

[78] T. van Ritbergen, S.A. Larin & J.A.M. Vermaseren, Phys. Lett. **B400** (1997), 379.

[79] H. Verschelde, S. Schelstraete & M. Vanderkelen, Z. Phys. **C76** (1997), 161.

[80] H. Verschelde, Phys. Lett. **B351** (1995), 242. Phys. Lett. **B516** (2001), 307.

[81] R.E. Browne & J.A. Gracey, JHEP **11** (2003), 029.

[82] D. Dudal, H. Verschelde, J.A. Gracey, V.E.R. Lemes, M.S. Sarandy, R.F. Sobreiro, S.P. Sorella, JHEP **0104** (2004), 044.

[83] D. Dudal, Phys. Lett. **B677** (2009), 203.

[84] A. Cucchieri & T. Mendes, Proc. Sci. LAT2007 (2007), 297; arXiv:0710.0412, [hep-lat].

[85] I.L. Bogolubsky, E.M. Ilgenfritz, M. Muller-Preussker & A. Sternbeck, Proc. Sci. LAT2007 (2007), 290; arXiv:0710.1968, [hep-lat].

[86] A.C. Aguilar, D. Binosi & J. Papavassiliou, Phys. Rev. **D78** (2008), 025010.

[87] A. Cucchieri & T. Mendes, Phys. Rev. Lett. **100** (2008), 241601.

[88] A. Cucchieri & T. Mendes, Phys. Rev. **D78** (2008), 094503.

[89] Ph. Boucaud, J.P. Leroy, A.L. Yaounac, J. Micheli, O. Pene & J. Rodriguez-Quintero, JHEP **06** (2008), 099.

[90] D. Dudal, S.P. Sorella, N. Vandersickel, H. Verschelde, Phys. Rev. **D77** (2008), 071501(R).

[91] D. Dudal, J.A. Gracey, S.P. Sorella, N. Vandersickel & H. Verschelde, Phys. Rev. **D78** (2008), 065047.